CSE 311 : Midterm Exam
Released Tuesday, February 16, 12:01 AM
Due Thursday, February 18, 11:59 PM

Notes

- This exam was designed to take you 2-3 hours to solve and write by hand, though you may work longer if you wish.
- If you wish to typeset it, that will take additional time.
- Problems 1b and 4 are marked with *, which is meant to indicate that those are harder problems, so don't be discouraged if you find them tricky or you get stuck. Take a little walk, think about something else, and come back.
- The extra credit problem will be worth more substantial points than the usual homework extra credit, worth between $.5$ and 1 of the other problems.

1. All the P's and all the Q's [10 points]

(a) Choose a domain of discourse and definitions of the predicates $P(\cdot,\cdot)$ and $Q(\cdot)$ so that the following statement is true, and give a short English proof that your choice works.

$$\exists x\, \big((\forall y\, P(x,y)) \leftrightarrow Q(x)\big) \wedge \big(Q(x) \to \exists y\, P(x,y)\big)$$

(b) * Give a formal symbolic proof of the following statement.

$$\forall x\, \big((\forall y\, P(x,y)) \leftrightarrow Q(x)\big) \to \big(Q(x) \to \exists y\, P(x,y)\big)$$

2. There's nothing there [10 points]

Let A be the set of prime numbers larger than 45 and let B be $\{k \in \mathbb{N} : \exists n(n > 2 \wedge k = n!)\}$.

Show that $A \cap B = \varnothing$ by contradiction.

3. A useful inequality [10 points]

Let $x > -1$ be a real number and $n \geq 0$ be an integer. Use induction to show that

$$(1+x)^n \geq 1 + nx.$$

4. A smaller set* [10 points]

Definition. For sets X and Y, $X \subsetneq Y$ means that X is a subset of Y but $X \neq Y$.
Definition. For any set X, $|X|$ is the number of elements in it.

Suppose $|A| = n$ and $B \subsetneq A$. Show, using induction, that there exists $k < n$ such that $|B| = k$. *Note: we know that this can be shown without induction, but the trick here is to show it by induction. Hint: think about which variable you should do induction upon, and reall that you cannot do induction on an existentially quantified variable.*

5. Decimals divided by three [10 points]

Fix an integer x for which $3 \mid x$. Consider the representation of x as a string of digits $n_k \ldots n_2 n_1$, where $n_i \in \{0, 1, \ldots, 9\}$ for all $i \in \{1, ..., k\}$, e.g. that n_1 refers to the ones digit, n_2 the tens, n_3 the hundreds, and so on. For example, 1,326 is represented as $n_4 = 1, n_3 = 3, n_2 = 2, n_1 = 6$ and $3|1,326$ as well as $3|12 = 1 + 3 + 2 + 6$. Prove that $3 \mid \sum_{i=1}^{k} n_i$.

6. Extra Credit: An i for an i

Fix a positive real number i for which $i + \frac{1}{i}$ is an integer. Argue, with proof, whether $i^{311} + \frac{1}{i^{311}}$ is an integer.