Notes

- This exam was designed to take you 2-3 hours to solve and write by hand, though you may work longer if you wish.
- If you wish to typeset it, that will take additional time.
- Problems 1b and 4 are marked with *, which is meant to indicate that those are harder problems, so don't be discouraged if you find them tricky or you get stuck. Take a little walk, think about something else, and come back.
- The extra credit problem will be worth more substantial points than the usual homework extra credit, worth between \( \frac{5}{6} \) and 1 of the other problems.

1. All the P’s and all the Q’s [10 points]

(a) Choose a domain of discourse and definitions of the predicates \( P(x, y) \) and \( Q(x) \) so that the following statement is true, and give a short English proof that your choice works.

\[
\exists x \left( (\forall y P(x, y)) \leftrightarrow Q(x) \right) \land (Q(x) \rightarrow \exists y P(x, y))
\]

(b) * Give a formal symbolic proof of the following statement.

\[
\forall x \left( (\forall y P(x, y)) \leftrightarrow Q(x) \right) \rightarrow (Q(x) \rightarrow \exists y P(x, y))
\]

2. There’s nothing there [10 points]

Let \( A \) be the set of prime numbers larger than 45 and let \( B = \{ k \in \mathbb{N} : \exists n (n > 2 \land k = n!) \} \).

Show that \( A \cap B = \emptyset \) by contradiction.

3. A useful inequality [10 points]

Let \( x > -1 \) be a real number and \( n \geq 0 \) be an integer. Use induction to show that

\[
(1 + x)^n \geq 1 + nx.
\]

4. A smaller set* [10 points]

**Definition.** For sets \( X \) and \( Y \), \( X \subseteq Y \) means that \( X \) is a subset of \( Y \) but \( X \neq Y \).

**Definition.** For any set \( X \), \( |X| \) is the number of elements in it.

Suppose \( |A| = n \) and \( B \subseteq A \). Show, using induction, that there exists \( k < n \) such that \( |B| = k \). Note: we know that this can be shown without induction, but the trick here is to show it by induction. Hint: think about which variable you should do induction upon, and recall that you cannot do induction on an existentially quantified variable.
5. Decimals divided by three [10 points]

Fix an integer $x$ for which $3 \mid x$. Consider the representation of $x$ as a string of digits $n_k \ldots n_2 n_1$, where $n_i \in \{0, 1, \ldots, 9\}$ for all $i \in \{1, \ldots, k\}$, e.g. that $n_1$ refers to the ones digit, $n_2$ the tens, $n_3$ the hundreds, and so on. For example, 1,326 is represented as $n_4 = 1, n_3 = 3, n_2 = 2, n_1 = 6$ and $3 \mid 1, 326$ as well as $3 \mid 12 = 1 + 3 + 2 + 6$. Prove that $3 \mid \sum_{i=1}^{k} n_i$.

6. Extra Credit: An $i$ for an $i$

Fix a positive real number $i$ for which $i + \frac{1}{i}$ is an integer. Argue, with proof, whether $i^{311} + \frac{1}{i^{311}}$ is an integer.