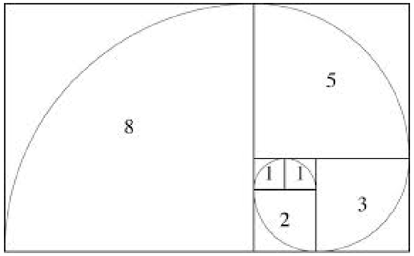
CSE 311: Foundations of Computing

Lecture 19: Structural induction

Image from @Hevesh5 on YouTube

Fibonacci Numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$ for all $n \geq 2$



Last lecture: $f_n < 2^n$ for all $n \geq 0$

Similar: $f_n \geq 2^{\frac{n}{2}-1}$ for all $n \geq 2$

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

This implies: $n \leq 1 + 2\log_2 a$

i.e., # of steps ≤ 1 + twice the # of bits in a.

Recap: Recursive Definitions of Sets

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

Example: The set Σ* of *strings* over the alphabet Σ is defined by

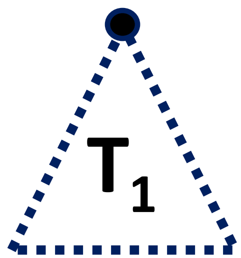
- **Basis:** $\mathcal{E} \in \Sigma$ ($\mathcal{E}$ is the empty string)
- **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

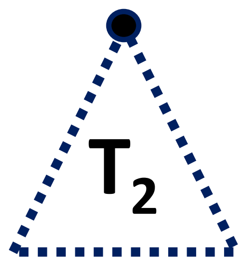
Example of function for recursively defined set: String length

- len($\mathcal{E}$) = 0
- len(wa) = 1 + len(w) for $w \in \Sigma^*$, $a \in \Sigma$

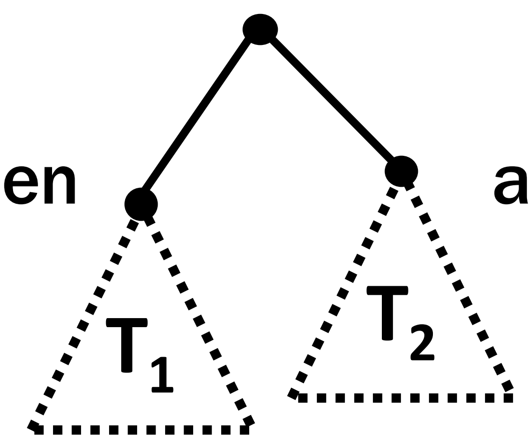
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

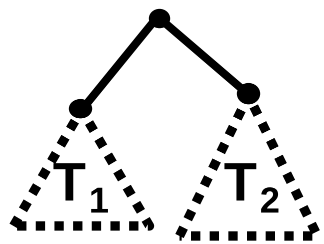
and 

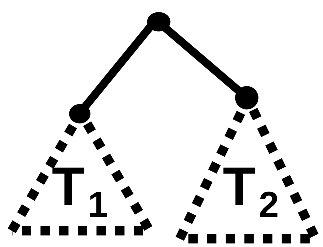
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)
- height(•) = 0
- height (

) = 1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

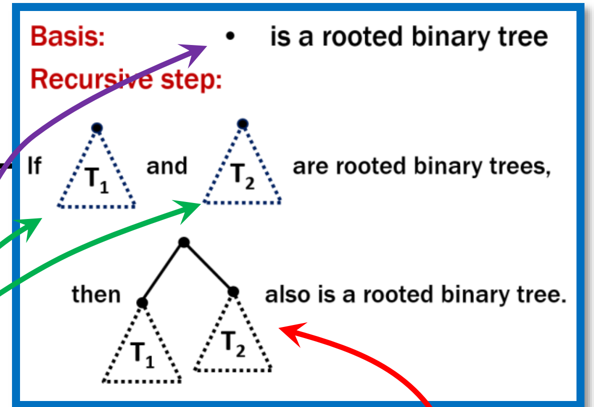
Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction



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Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

Using Structural Induction

- Let S be given by…
  - **Basis:** $6 \in S$; $15 \in S$;
  - **Recursive:** if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

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1. Let P(x) be "3|x". We prove that P(x) is true for all x ∈ S by structural induction.

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2. Base Case: 3|6 and 3|15 so P(6) and P(15) are true

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   Since P(x) is true, 3|x and so x=3m for some integer m and since P(y) is true, 3|y and so y=3n for some integer n. Therefore x+y=3m+3n=3(m+n) and thus 3|(x+y).

   Hence P(x+y) is true.
5. Therefore by induction 3|x for all x ∈ S.

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Recursive: if $x, y \in S$ then $x + y \in S$

Lecture 19 Activity

You will be assigned to breakout rooms. Please:

- Introduce yourself
- Choose someone to share their screen, showing this PDF
- Recall that we defined recursively
  - Strings. Basis: $\varepsilon \in \Sigma^*$ Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$
  - Concatenation of strings. Basis: $x \cdot \varepsilon = x$ for Σ^*
    Recursive: $x \cdot wa = (x \cdot w)a$ for $x \in \Sigma^*$, $a \in \Sigma$
- Let $P(y)$ be ``$\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$''
- We want to prove $P(y)$ fo all $y \in \Sigma^*$ by structural induction. Please complete the proof:

Base Case: Let $x \in \Sigma^*$ arbitrary. Then $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + 0 = \text{len}(x) + \text{len}(\varepsilon)$. Hence $P(\varepsilon)$.

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary $w \in \Sigma^*$.

Inductive Step: ...

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- Let $P(y)$ be ``$\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$''
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Inductive Hypothesis: Suppose $P(w)$ for an arbitrary $w \in \Sigma^*$.

Inductive Step: Let a ∈ Σ. Let x ∈ Σ*. Then len(x • wa) = len((x • w)a) by defn of •

= len(x • w)+1 by defn of len

= len(x)+len(w)+1 by I.H.

= len(x)+len(wa) by defn of len

Therefore len(x • wa)= len(x)+len(wa) for all x ∈ Σ*, so P(wa) is true.

So, by induction len(x • y) = len(x) + len(y) for all x,y ∈ Σ*

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1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \le 2^{\text{height}(T)+1} - 1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**

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2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$, **and** $2^{0+1}-1=2^1-1=1$ **so** $P(\bullet)$ **is true.**

Claim: For every rooted binary tree T, $\text{size}(T) \leq 2^{\text{height}(T) + 1} - 1$

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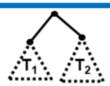
4. Inductive Step: **Goal: Prove P().**

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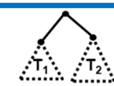
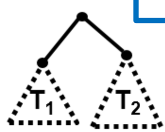
Goal: Prove P().

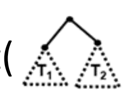
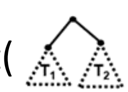
By defn, size() $= 1 + \text{size}(T_1) + \text{size}(T_2)$

$\leq 1 + 2^{\text{height}(T_1)+1} - 1 + 2^{\text{height}(T_2)+1} - 1$

by IH for T_1 and T_2

$\leq 2^{\text{height}(T_1)+1} + 2^{\text{height}(T_2)+1} - 1$

$\leq 2(2^{\max(\text{height}(T_1),\text{height}(T_2))+1}) - 1$

$\leq 2(2^{\text{height}(}$  $^{)}) - 1 \leq 2^{\text{height}(}$  $^{)+1} - 1$

which is what we wanted to show.

5. So, the P(T) is true for all rooted bin. trees by structural induction.