CSE 311: Foundations of Computing

Lecture 19: Structural induction

Image from @Hevesh5 on YouTube
Fibonacci Numbers

\[ f_0 = 0 \quad \text{gcd}(a, b) = \text{gcd}(b, a - b) \]
\[ f_1 = 1 \]
\[ f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2 \]

Last lecture: \( f_n < 2^n \) for all \( n \geq 0 \)

Similar: \( f_n \geq \frac{n}{2} - 1 \) for all \( n \geq 2 \)

**Theorem:** Suppose that Euclid’s Algorithm takes \( n \) steps for \( \text{gcd}(a, b) \) with \( a \geq b > 0 \). Then, \( a \geq f_{n+1} \).

This implies: \( n \leq 1 + 2\log_2 a \)

i.e., # of steps \( \leq 1 + \) twice the # of bits in \( a \).
Recap: Recursive Definitions of Sets

Recursive definition

- **Basis step:** Some specific elements are in $S$
- **Recursive step:** Given some existing named elements in $S$, some new objects constructed from these named elements are also in $S$.
- **Exclusion rule:** Every element in $S$ follows from basis steps and a finite number of recursive steps

Example: The set $\Sigma^*$ of **strings** over the alphabet $\Sigma$ is defined by

- **Basis:** $\varepsilon \in \Sigma^*$ ($\varepsilon$ is the empty string)
- **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Example of function for recursively defined set: **String length**

- $\text{len}(\varepsilon) = 0$
- $\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$
Rooted Binary Trees

• **Basis:** is a rooted binary tree

• **Recursive step:**

\[ \text{Tree}(L_1, R_1) \quad \text{Tree}(L_2, R_2) \]

If \( T_1 \) and \( T_2 \) are rooted binary trees,

\[ \text{Tree} \left( \text{Tree} \left( L_1, R_1 \right), \text{Tree} \left( L_2, R_2 \right) \right) \]

then also is a rooted binary tree.
Defining Functions on Rooted Binary Trees

• $\text{size}(\bullet) = 1$

• $\text{size} \left( \begin{array}{c} T_1 \\ \bullet \\ T_2 \end{array} \right) = 1 + \text{size}(T_1) + \text{size}(T_2)$

• $\text{height}(\bullet) = 0$

• $\text{height} \left( \begin{array}{c} T_1 \\ \bullet \\ T_2 \end{array} \right) = 1 + \max\{\text{height}(T_1), \text{height}(T_2)\}$
Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all specific elements $u$ of $S$ mentioned in the *Basis step*.

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*.

**Inductive Step:** Prove that $P(w)$ holds for each of the new elements $w$ constructed in the *Recursive step* using the named elements mentioned in the *Inductive Hypothesis*.

**Conclude** that $\forall x \in S, P(x)$.
Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all specific elements $u$ of $S$ mentioned in the Basis step

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of each of the existing named elements mentioned in the Recursive step

**Inductive Step:** Prove that $P(w)$ holds for each of the new elements $w$ constructed in the Recursive step using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall x \in S, P(x)$
Using Structural Induction

• Let $S$ be given by...
  – **Basis:** $6 \in S$; $15 \in S$;
  – **Recursive:** if $x, y \in S$ then $x + y \in S$.

**Claim:** Every element of $S$ is divisible by 3.
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1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

   Basis: $6 \in S$; $15 \in S$;
   Recursive: if $x, y \in S$ then $x + y \in S$
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \vert x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \vert 6$ and $3 \vert 15$ so $P(6)$ and $P(15)$ are true

3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x,y \in S$

4. Inductive Step: **Goal:** Show $P(x+y)$

   Basis: $6 \in S$; $15 \in S$;

   Recursive: if $x,y \in S$ then $x + y \in S$
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

4. Inductive Step: Goal: Show $P(x+y)$.

   Since $P(x)$ is true, $3 \mid x$ and so $x = 3m$ for some integer $m$ and since $P(y)$ is true, $3 \mid y$ and so $y = 3n$ for some integer $n$. Therefore $x + y = 3m + 3n = 3(m+n)$ and thus $3 \mid (x+y)$.

   Hence $P(x+y)$ is true.

5. Therefore by induction $3 \mid x$ for all $x \in S$.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$. 
Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be “for all $x \in S$ that can be constructed in at most $n$ recursive steps, $P(x)$ is true.”
You will be assigned to **breakout rooms**. Please:

- Introduce yourself
- Choose someone to share their screen, showing this PDF
- Recall that we defined recursively
  - **Strings.** Basis: $\varepsilon \in \Sigma^*$ Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$
  - **Concatenation of strings.** Basis: $x \cdot \varepsilon = x$ for $\Sigma^*$
    Recursive: $x \cdot wa = (x \cdot w)a$ for $x \in \Sigma^*$, $a \in \Sigma$
- Let $P(y)$ be ``$\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma$’’
- We want to prove $P(y)$ for all $y \in \Sigma^*$ by **structural induction**. Please complete the proof:

  **Base Case:** Let $x \in \Sigma^*$ arbitrary. Then $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + 0 = \text{len}(x) + \text{len}(\varepsilon)$. Hence $P(\varepsilon)$.

  **Inductive Hypothesis:** Suppose $P(w)$ for an arbitrary $w \in \Sigma^*$.

  **Inductive Step:** ...

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Go to [pollev.com/philipmg](http://pollev.com/philipmg) and login with your UW identity.
Recall that we defined recursively:

- **Strings.** Basis: \( \varepsilon \in \Sigma^* \) Recursive: If \( w \in \Sigma^* \) and \( a \in \Sigma \) then \( wa \in \Sigma^* \)
- **Concatenation of strings.** Basis: \( x \cdot \varepsilon = x \) for \( \Sigma^* \)
  Recursive: \( x \cdot wa = (x \cdot w)a \) for \( x \in \Sigma^* \), \( a \in \Sigma \)

Let \( P(y) \) be "\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)"

We want to prove \( P(y) \) for all \( y \in \Sigma^* \) by **structural induction**. Please complete the proof:

**Base Case:** Let \( x \in \Sigma^* \) arbitrary. Then \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + 0 = \text{len}(x) + \text{len}(\varepsilon) \). Hence \( P(\varepsilon) \).

**Inductive Hypothesis:** Suppose \( P(w) \) for an arbitrary \( w \in \Sigma^* \).

**Inductive Step:** Let \( a \in \Sigma \). Let \( x \in \Sigma^* \). Then \( \text{len}(x \cdot wa) = \text{len}((x \cdot w)a) \) by defn of \( \cdot \)

\[
= \text{len}(x \cdot w) + 1 \quad \text{by defn of len}
= \text{len}(x) + \text{len}(w) + 1 \quad \text{by I.H.}
= \text{len}(x) + \text{len}(wa) \quad \text{by defn of len}
\]

Therefore \( \text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa) \) for all \( x \in \Sigma^* \), so \( P(wa) \) is true.

So, by induction \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)
Claim: For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$
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1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.
Claim: For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$

1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.

2. Base Case: $\text{size}(\bullet) = 1$, $\text{height}(\bullet) = 0$, and $2^{0+1} - 1 = 2^1 - 1 = 1$ so $P(\bullet)$ is true.
Claim: For every rooted binary tree \( T \), \( \text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1 \)

1. Let \( P(T) \) be “\( \text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1 \)”. We prove \( P(T) \) for all rooted binary trees \( T \) by structural induction.

2. Base Case: \( \text{size}(\bullet) = 1 \), \( \text{height}(\bullet) = 0 \), and \( 2^{0+1} - 1 = 2^1 - 1 = 1 \) so \( P(\bullet) \) is true.

3. Inductive Hypothesis: Suppose that \( P(T_1) \) and \( P(T_2) \) are true for some rooted binary trees \( T_1 \) and \( T_2 \).

4. Inductive Step: Goal: Prove \( P(\text{rooted binary tree}) \).
**Claim:** For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$

1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)}+1-1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.

2. Base Case: $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$, and $2^{0+1}-1=2^1-1=1$ so $P(\bullet)$ is true.

3. Inductive Hypothesis: Suppose that $P(T_1)$ and $P(T_2)$ are true for some rooted binary trees $T_1$ and $T_2$.

4. Inductive Step: 
   
   **Goal:** Prove $P(T)$.

   By defn, $\text{size}(T) = 1 + \text{size}(T_1) + \text{size}(T_2)$
   
   $\leq 1 + 2^{\text{height}(T_1)+1-1} + 2^{\text{height}(T_2)+1-1}$
   
   by IH for $T_1$ and $T_2$
   
   $\leq 2^{\text{height}(T_1)+1} + 2^{\text{height}(T_2)+1-1}$
   
   $\leq 2(2^{\text{max}(\text{height}(T_1),\text{height}(T_2))}+1)-1$
   
   $\leq 2(2^{\text{height}(T)})-1 \leq 2^{\text{height}(T)+1} - 1$
   
   which is what we wanted to show.

5. So, the $P(T)$ is true for all rooted bin. trees by structural induction.