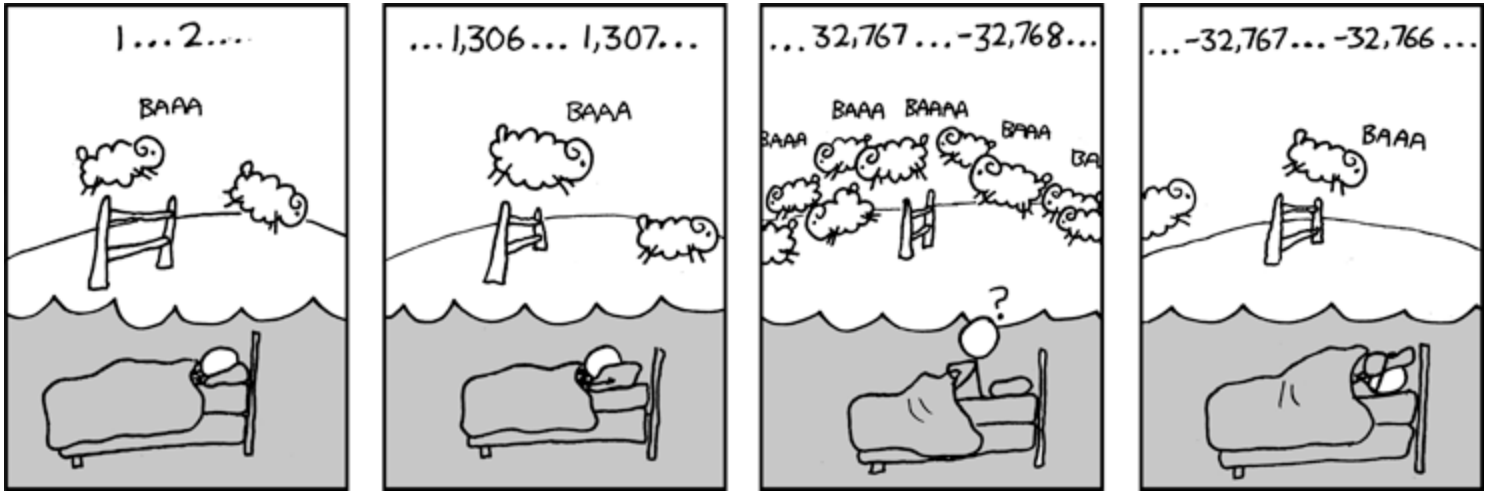
CSE 311: Foundations of Computing

Lecture 15: The Euclidean algorithm and applications



Recap from last lecture

- An integer $p > 2$ is prime if the only positive factors are 1 and p
- A prime factorization for an integer $n > 1$ is of the form $n = p_1 \cdot \ldots \cdot p_k$ where $p_1, \ldots, p_k$ are prime numbers.
- Each integer $n > 1$ as a unique prime factorization.
- The greatest common divisor $\gcd(a, b)$ is the largest integer d with $d \mid a$ and $d \mid b$.
- Important fact for today: $\gcd(a, b) = \gcd(b, a \,\%\, b)$

Another simple GCD fact

If a is a positive integer, gcd(a,0) = a.

Euclid's Algorithm

gcd(a, b) = gcd(b, a mod b), gcd(a,0)=a

```
int gcd(int a, int b){ /* a >= b, b >= 0 */
    if (b == 0) {
        return a;
    }
    else {
        return gcd(b, a % b);
    }
```

Example: GCD(660, 126)

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) =

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \,\%\, b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) = gcd(126, 660 % 126) = gcd(126, 30)
= gcd(30, 126 % 30) = gcd(30, 6)
= gcd(6, 30 % 6) = gcd(6, 0)
= 6

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) = gcd(126, 660 % 126) = gcd(126, 30)
= gcd(30, 126 % 30) = gcd(30, 6)
= gcd(6, 30 % 6) = gcd(6, 0)
= 6

In tableau form:

$a = q \cdot d + r$

660 = 5 * 126 + 30
126 = 4 * 30 + 6
30 = 5 * 6 + 0

660 % 126 = 30
126 % 30 = 6
30 % 6 = 0

Bézout's theorem

If a and b are positive integers, then there exist integers $\boldsymbol{s}$ and $\boldsymbol{t}$ such that

$$\gcd(a,b) = \boldsymbol{s}a + \boldsymbol{t}b.$$

Proof is algorithmic: via the Extended Euclidean algorithm

Extended Euclidean algorithm

- **Method:** $\text{extGCD}(a, b) \rightarrow (g, s, t)$
- **Input: Integers** $a \geq b \geq 0$
- **Output: Integers** g, s, t **s.t.** $g = \gcd(a, b) = s \cdot a + t \cdot b$

1. **IF** $b == 0$ **THEN return** $(a, 1, 0)$ $\quad // \; a = \gcd(a, 0) = 1 \cdot a + 0 \cdot 0$
2. $(g, s, t) \coloneqq \text{extGCD}(b, a \,\%\, b)$
3. **Write** $a = qb + (a \,\%\, b)$ **with** $q \in \mathbb{Z}$
4. **Return** $(g, t, s - tq)$

$// \; g = \gcd(b, a \,\%\, b) = \gcd(a, b)$
$// \; g = s \cdot b + t \cdot (a \,\%\, b)$
$// \; = s \cdot b + t \cdot (a - qb)$
$// \; = t \cdot a + (s - tq) \cdot b$

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

Extended Euclidean algorithm

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  extGCD(31, 16) -> extGCD(16, 31 % 16)

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

extGCD(15, 1) -> extGCD(1, 15 % 1)

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)
extGCD(16, 15) -> extGCD(15, 16 % 15)
extGCD(15, 1) -> extGCD(1, 15 % 1)
extGCD(1, 0) = 1 = 1*1 + 0*0

Hence $\gcd(31,16) = 1$

Extended Euclidean algorithm

- **Example: Find s, t such that $\gcd(31,16) = s \cdot 31 + t \cdot 16$**

extGCD(31, 16) -> extGCD(16, 31 % 16)

extGCD(16, 15) -> extGCD(15, 16 % 15)

extGCD(15, 1) -> extGCD(1, 15 % 1)

extGCD(1, 0) = 1 = 1*1 + 0*0

$s \cdot 1 + t \cdot 0$

15=15*1 + 0

1 = 0*15 + (1 - 0*15)*1 = 0*15 + 1*1

Hence $\gcd(31,16) = 1$

Extended Euclidean algorithm

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extGCD(15, 1) -> extGCD(1, 15 % 1)

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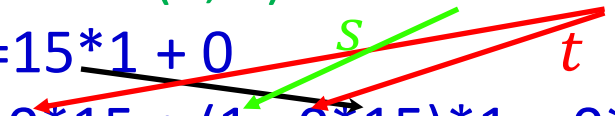

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extGCD(16, 15) -> extGCD(15, 16 % 15)

extGCD(15, 1) -> extGCD(1, 15 % 1)

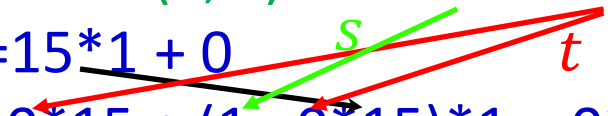
extGCD(1, 0) = 1 = 1*1 + 0*0

15=15*1 + 0

1 = 0*15 + (1 - 0*15)*1 = 0*15 + 1*1

16=1*15 + 1

1 = 1*16 + (0 - 1*1)*15 = 1*16 + -1*15



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extGCD(1, 0) = 1 = 1*1 + 0*0

15=15*1 + 0

1 = 0*15 + (1 - 0*15)*1 = 0*15 + 1*1

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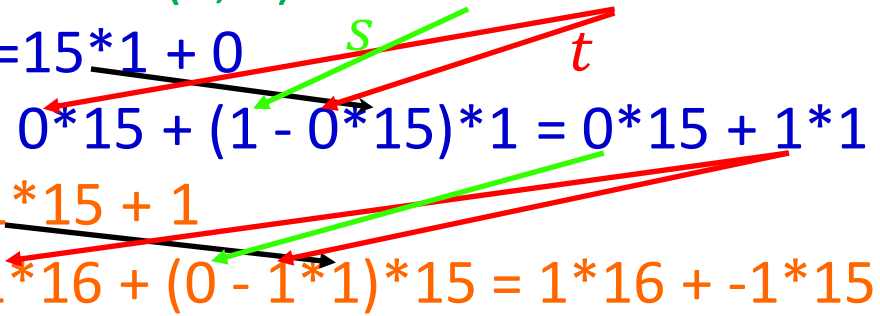

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extGCD(1, 0) = 1 = 1*1 + 0*0

15=15*1 + 0

1 = 0*15 + (1 - 0*15)*1 = 0*15 + 1*1

16=1*15 + 1

1 = 1*16 + (0 - 1*1)*15 = 1*16 + -1*15

31=1*16 + 15

1 = -1*31 + (1 - -1*1)*16 = -1*31 + 2*16

s t

s t

$1 = 1 \cdot 16 + (-1) 15$

$15 = 31 - 1 \cdot 16$

$1 = 1 \cdot 16 + (-1)(31 - 1 \cdot 16)$

Hence $\gcd(31,16) = 1 = (-1) \cdot 31 + 2 \cdot 16$

Extended Euclidean algorithm

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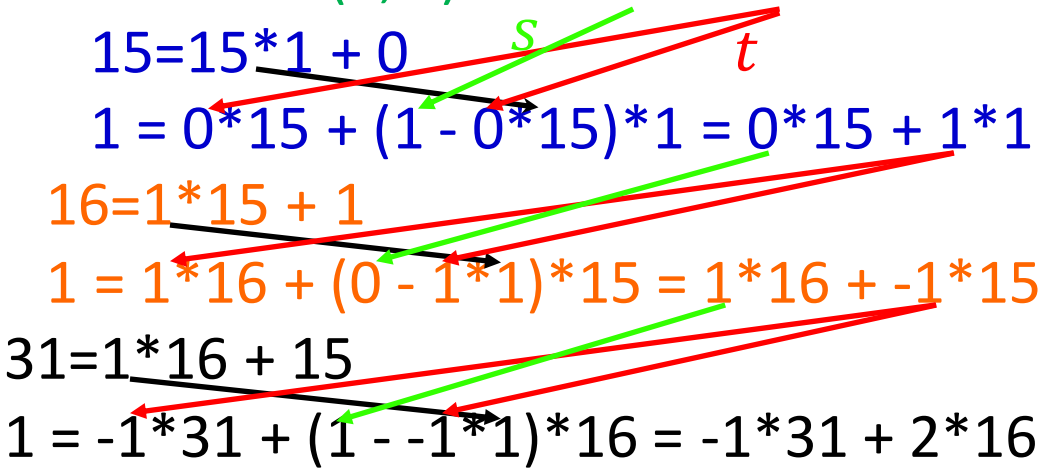

Hence $\gcd(31,16) = 1 = (-1) \cdot 31 + 2 \cdot 16$

Extended Euclidean algorithm

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extGCD(31, 16) -> extGCD(16, 31 % 16)
  extGCD(16, 15) -> extGCD(15, 16 % 15)
    extGCD(15, 1) -> extGCD(1, 15 % 1)
      extGCD(1, 0) = 1 = **1***1 + **0***0
    15=15*1 + 0
    1 = **0***15 + **(1 - 0*15)***1 = **0***15 + **1***1
  16=1*15 + 1
  1 = **1***16 + **(0 - 1*1)***15 = **1***16 + -**1***15
31=1*16 + 15
1 = **-1***31 + **(1 - -1*1)***16 = **-1***31 + **2***16

Hence $\gcd(31,16) = 1 = (-1) \cdot 31 + 2 \cdot 16$

Multiplicative inverse $\mathrm{mod}\ m$

Suppose $\mathrm{GCD}(a, m) = 1$

By Bézout's Theorem, there exist integers s and t such that $sa + tm = 1$.

$1 \% m = sa \% m$

$1 \equiv sa \pmod{m}$

$s \% m$ is the multiplicative inverse of a:

$sa \equiv 1 \pmod{m}$

$1 = (sa + tm) \% m = sa \% m$

$sa \equiv 1 \pmod{m}$

$sa \cdot 5 \equiv 5 \pmod{m}$

Suppose $a \cdot x \equiv 1 \pmod{m}$

then $sa \cdot x \equiv 5 \pmod{m}$

$1 \cdot x \equiv 5 \pmod{m}$

$2x \equiv 2y \leftarrow x \equiv y \pmod{m}$

Multiplicative inverse $\bmod m$

Suppose $\mathrm{GCD}(a, m) = 1$

By Bézout's Theorem, there exist integers s and t such that $sa + tm = 1$.

$s \,\%\, m$ is the multiplicative inverse of a:

$$1 = (sa + tm) \,\%\, m = sa \,\%\, m$$

Example

Solve: $7x \equiv 1 \pmod{26}$

$x = \frac{1}{7}$ No fractions just integers

$7x \equiv 5 \pmod{26}$ $7 \equiv 1 \pmod{26}$

$x = 1$ $x \equiv 5 \pmod{26}$

Example

$8x \equiv 1 \pmod{26}$

i.e. $8x \equiv 2 \pmod{26}$

$\Rightarrow 8x \equiv 5 \pmod{26}$

Solve: $7x \equiv 1 \pmod{26}$ $\quad 2x \equiv 5$

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$a = d \cdot q + r$ $\qquad r = a - d \cdot q$

$$26 = 7 * 3 + 5 \qquad 5 = 26 - 7 * 3$$

$$7 = 5 * 1 + 2 \qquad 2 = 7 - 5 * 1$$

$$5 = 2 * 2 + 1 \qquad 1 = 5 - 2 * 2$$

$$\begin{aligned} 1 &= 5 - 2 * (7 - 5 * 1) \quad (2) \\ &= (-7) * 2 + 3 * 5 \\ &= (-7) * 2 + 3 * (26 - 7 * 3) \quad (5) \\ &= (-11) * 7 + 3 * 26 \end{aligned}$$

$7 \cdot (-11) = -77$

$7 \cdot 15 = 105 = 4 \cdot 26 + 1$

Multiplicative inverse of 7 mod 26

Now $(-11) \bmod 26 = 15$. **So,** $x = 15 + 26k$ **for** $k \in \mathbb{Z}$.

Example of a more general equation

Now solve: $7y \equiv 3 \pmod{26}$

We already computed that 15 **is the multiplicative inverse of** 7 **modulo** 26:

That is, $7 \cdot 15 \equiv 1 \pmod{26}$

By the multiplicative property of mod we have

$$7 \cdot 15 \cdot 3 \equiv 3 \pmod{26}$$

So any $y \equiv 15 \cdot 3 \pmod{26}$ **is a solution.**

That is, $y = 19 + 26k$ **for any integer** k **is a solution.**

Lecture 15 Activity

You will be assigned to breakout rooms. Please:

- Introduce yourself
- Choose someone to share their screen, showing this PDF
- Discuss the following questions:

1. If you run the extended Euclidean algorithm for $(51,23)$ it will return that $\gcd(51,23) = 1 = (-9) \cdot 51 + 20 \cdot 23$. What is this telling you about the multiplicative inverse of 51 modulo 23.
2. If you run the extended Euclidean algorithm for $(51,24)$ it will return that $\gcd(51,24) = 3 = 1 \cdot 51 + (-2) \cdot 24$. What is this telling you about the multiplicative inverse of 51 modulo 24.
3. What is the set of integers that do not have a multiplicative inverse modulo 10?

Fill out the poll everywhere for Activity Credit!
Go to pollev.com/philipmg and login with your UW identity

Lecture 15 Activity

1. If you run the extended Euclidean algorithm for $(51,23)$ it will return that $\gcd(51,23) = 1 = (-9) \cdot 51 + 20 \cdot 23$. What is this telling you about the multiplicative inverse of 51 modulo 23.
   Solution: $-9 + 23 = 14$
2. If you run the extended Euclidean algorithm for $(51,24)$ it will return that $\gcd(51,24) = 3 = 1 \cdot 51 + (-2) \cdot 24$. What is this telling you about the multiplicative inverse of 51 modulo 24.
   Solution: There is none.
3. What is the set of integers that do not have a multiplicative inverse modulo 10?
   Solution: $0 + 10x, 2 + 20x, 5 + 10x, x \in \mathbb{Z}$

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Math mod a prime is especially nice

$\gcd(a, m) = 1$ **if** m **is prime and** $0 < a < m$ **so can always solve these equations mod a prime.**

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

mod 7

Modular Exponentiation % 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

Exponentiation

- **Compute** 78365^{81453}

- **Compute** $78365^{81453} \% 104729$

- **Output is small**
  - need to keep intermediate results small

Repeated Squaring – small and fast

Since $a \,\%\, m \equiv a \pmod{m}$ and $b \,\%\, m \equiv b \pmod{m}$

we have $ab \,\%\, m = \big((a \,\%\, m)(b \,\%\, m)\big) \%\, m$

So $\quad a^2 \,\%\, m \;=\; (a \,\%\, m)^2 \,\%\, m$

and $\quad a^4 \,\%\, m \;=\; (a^2 \,\%\, m)^2 \,\%\, m$

and $\quad a^8 \,\%\, m \;=\; (a^4 \,\%\, m)^2 \,\%\, m$

and $\quad a^{16} \,\%\, m \;=\; (a^8 \,\%\, m)^2 \,\%\, m$

and $\quad a^{32} \,\%\, m \;=\; (a^{16} \,\%\, m)^2 \,\%\, m$

Can compute $a^k \,\%\, m$ for $k = 2^i$ in only i steps

What if k is not a power of 2?

Fast Exponentiation Algorithm

81453 in binary is 10011111000101101

$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$

$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$

$a^{81453} \% m =$
$(\ldots(((((a^{2^{16}} \% m \cdot$
$a^{2^{13}} \% m) \% m \cdot$
$a^{2^{12}} \% m) \% m \cdot$
$a^{2^{11}} \% m) \% m \cdot$
$a^{2^{10}} \% m) \% m \cdot$
$a^{2^{9}} \% m) \% m \cdot$
$a^{2^{5}} \% m) \% m \cdot$
$a^{2^{3}} \% m) \% m \cdot$
$a^{2^{2}} \% m) \% m \cdot$
$a^{2^{0}} \% m) \% m$

The fast exponentiation algorithm computes $a^k \% m$ using $\leq 2\log k$ multiplications $\% m$

Fast Exponentiation: $a^k \,\%\, m$ for all k

Another way....

$$a^{2j} \% \, m = \left(a^j \% \, m\right)^2 \% \, m$$

$$a^{2j+1} \% \, m \; = \; \left((a \% \, m) \cdot (a^{2j} \% \, m)\right) \% \, m$$

Fast Exponentiation

```
public static long FastModExp(long a, long k, long modulus) {
        long result = 1;
        long temp;

        if (k > 0) {
            if ((k % 2) == 0) {
                temp = FastModExp(a,k/2,modulus);
                result = (temp * temp) % modulus;
            }
            else {
                temp = FastModExp(a,k-1,modulus);
                result = (a * temp) % modulus;
            }
        }
        return result;
    }
```

$$a^{2j} \% \, m = \left(a^{j} \% \, m\right)^{2} \% \, m$$

$$a^{2j+1} \% \, m = \left((a \% \, m) \cdot (a^{2j} \% \, m)\right) \% \, m$$

Using Fast Modular Exponentiation

- Your e-commerce web transactions use SSL (Secure Socket Layer) based on RSA encryption
- RSA
  - Vendor chooses random 512-bit or 1024-bit primes p, q and 512/1024-bit exponent e. Computes $m = p \cdot q$
  - Vendor broadcasts (m, e)
  - To send a to vendor, you compute $C = a^e \% m$ using *fast modular exponentiation* and send C to the vendor.
  - Using secret p, q the vendor computes d that is the *multiplicative inverse* of e mod $(p-1)(q-1)$.
  - Vendor computes $C^d \% m$ using *fast modular exponentiation*.
  - **Fact:** $a = C^d \% m$ for $0 < a < m$ unless $p|a$ or $q|a$