Asking homework questions on Ed

- New system on Ed: we are creating **threads** of the form “Homework X, Problem Y” for each homework problem.
- If you have a question for a particular homework problem, please:
  - Read the whole thread
  - Then ask the question in that thread

- Link: https://edstem.org/us/courses/4896/discussion/
Last class: Intro to predicate logic

- **Domain of discourse** = variable range
- **Predicates**: Functions $P(x)$ that return a truth value for each $x$ in domain (predicates may depend on more than one variable, e.g. $Q(x_1, x_2, x_3)$)
- **Existential quantor** $\exists$ and **universal quantor** $\forall$
Last class: Intro to predicate logic

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<th>Domain of Discourse</th>
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<tr>
<td>Positive Integers</td>
<td>Even(x) ::= “x is even” Greater(x, y) ::= “x &gt; y”</td>
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<tr>
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<td>Odd(x) ::= “x is odd”</td>
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Determine the truth values of each of these statements:

$\forall x \ (\text{Even}(x) \lor \text{Odd}(x))$

$\exists x \ (\text{Even}(x) \land \text{Odd}(x))$

$\forall x \ \text{Greater}(x+1, x)$
Last class: Intro to predicate logic

- **Domain of discourse** = variable range
- **Predicates**: Functions $P(x)$ that return a truth value for each $x$ in domain (predicates may depend on more than one variable, e.g. $Q(x_1, x_2, x_3)$)
- **Existential quantor** $\exists$ and **universal quantor** $\forall$

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Determine the truth values of each of these statements:

- $\forall x (\text{Even}(x) \lor \text{Odd}(x))$  \text{T}  every integer is either even or odd
- $\exists x (\text{Even}(x) \land \text{Odd}(x))$  \text{F}  no integer is both even and odd
- $\forall x \text{Greater}(x+1, x)$  \text{T}  adding 1 makes a bigger number
English to Predicate Logic

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<td>Mammals</td>
<td>Cat(x) ::= “x is a cat”</td>
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<tr>
<td></td>
<td>Red(x) ::= “x is red”</td>
</tr>
<tr>
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<td>LikesTofu(x) ::= “x likes tofu”</td>
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“Red cats like tofu”

“Some red cats don’t like tofu”
“Red cats like tofu”

\[ \forall x ((\text{Red}(x) \land \text{Cat}(x)) \rightarrow \text{LikesTofu}(x)) \]

“Some red cats don’t like tofu”

\[ \exists y ((\text{Red}(y) \land \text{Cat}(y)) \land \lnot \text{LikesTofu}(y)) \]
English to Predicate Logic

**Predicate Definitions**
- Cat(x) ::= “x is a cat”
- Red(x) ::= “x is red”
- LikesTofu(x) ::= “x likes tofu”

**Domain of Discourse**
- Mammals

When putting two predicates together like this, we use an “and”.

“**Red cats** like tofu”

When there’s no leading quantification, it means “for all”.

“**Some red cats** don’t like tofu”

When restricting to a smaller domain in a “for all” we use implication.

When restricting to a smaller domain in an “exists” we use and.

“**Some**” means “there exists”.

Negations of Quantifiers

Predicate Definitions

| PurpleFruit(x) ::= “x is a purple fruit” |

\[(*) \ \forall x \ \text{PurpleFruit}(x) \ (\text{“All fruits are purple”})\]

What is the negation of (*)?

(a) “there exists a purple fruit”
(b) “there exists a non-purple fruit”
(c) “all fruits are not purple”

Try your intuition! Which one “feels” right?

**Key Idea:** In every domain, exactly one of a statement and its negation should be true.
Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= “x is a purple fruit”

\( (*) \forall x \text{ PurpleFruit}(x) \) (“All fruits are purple”)

What is the negation of \((*)\)?
(a) “there exists a purple fruit”
(b) “there exists a non-purple fruit”
(c) “all fruits are not purple”

Key Idea: In every domain, exactly one of a statement and its negation should be true.
Negations of Quantifiers

Predicate Definitions

PurpleFruit(x) ::= “x is a purple fruit”

\((*)\) \(\forall x\) PurpleFruit(x) (”All fruits are purple”)

What is the negation of (\(*)\)?

(a) “there exists a purple fruit”
(b) “there exists a non-purple fruit”
(c) “all fruits are not purple”

Key Idea: In every domain, exactly one of a statement and its negation should be true.

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<tbody>
<tr>
<td>{plum}</td>
<td>{apple}</td>
<td>{plum, apple}</td>
</tr>
</tbody>
</table>

(*), (a) (b), (c) (a), (b)

The only choice that ensures exactly one of the statement and its negation is (b).
De Morgan’s Laws for Quantifiers

\[ \neg \forall x \ P(x) \equiv \exists x \ \neg P(x) \]
\[ \neg \exists x \ P(x) \equiv \forall x \ \neg P(x) \]
De Morgan’s Laws for Quantifiers

\[
\neg \forall x \; P(x) \equiv \exists x \; \neg P(x)
\]

\[
\neg \exists x \; P(x) \equiv \forall x \; \neg P(x)
\]

“There is no largest integer”

\[
\neg \exists x \; \forall y \; (x \geq y)
\]

\[
\equiv \forall x \; \neg \forall y \; (x \geq y)
\]

\[
\equiv \forall x \; \exists y \; (x \geq y)
\]

\[
\equiv \forall x \; \exists y \; (y > x)
\]

“For every integer, there is a larger integer”
Scope of Quantifiers

\[ \exists x \ (P(x) \land Q(x)) \quad \text{vs.} \quad \exists x \ P(x) \land \exists x \ Q(x) \]
scope of quantifiers

$$\exists x \ (P(x) \land Q(x))$$  vs.  $$\exists x \ P(x) \land \exists x \ Q(x)$$

This one asserts P and Q of the same $x$.

This one asserts P and Q of potentially different $x$’s.
Scope of Quantifiers

Example: \( \text{NotLargest}(x) \equiv \exists y \text{ Greater } (y, x) \equiv \exists z \text{ Greater } (z, x) \)

truth value:
- doesn’t depend on \( y \) or \( z \) “bound variables”
- does depend on \( x \) “free variable”

quantifiers only act on free variables of the formula they quantify

\[
\forall x (\exists y \ (P(x, y) \rightarrow \forall x \ Q(y, x)))
\]
Quantifier “Style”

This isn’t “wrong”, it’s just horrible style. Don’t confuse your reader by using the same variable multiple times...there are a lot of letters...

\[ \forall x (\exists y (P(x,y) \rightarrow \forall x Q(y, x))) \]
Nested Quantifiers

• **Bound variable names don’t matter**

\[ \forall x \exists y \ P(x, y) \equiv \forall a \exists b \ P(a, b) \]

• **Positions of quantifiers can sometimes change**

\[ \forall x (Q(x) \land \exists y \ P(x, y)) \equiv \forall x \exists y (Q(x) \land P(x, y)) \]

• **But: order is important...**
Quantifier Order Can Matter

Domain of Discourse
Integers
OR
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= “x ≥ y”

“There is a number greater than or equal to all numbers.”

∃x ∀y GreaterEq(x, y)))

“Every number has a number greater than or equal to it.”

∀y ∃x GreaterEq(x, y)))

The purple statement requires an entire row to be true.
The red statement requires one entry in each column to be true.
# Quantification with Two Variables

<table>
<thead>
<tr>
<th>expression</th>
<th>when true</th>
<th>when false</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \ \forall y \ P(x, y)$</td>
<td>Every pair is true.</td>
<td>At least one pair is false.</td>
</tr>
<tr>
<td>$\exists x \ \exists y \ P(x, y)$</td>
<td>At least one pair is true.</td>
<td>All pairs are false.</td>
</tr>
<tr>
<td>$\forall x \ \exists y \ P(x, y)$</td>
<td>We can find a specific $y$ for each $x$.</td>
<td>Some $x$ doesn’t have a corresponding $y$.</td>
</tr>
<tr>
<td>$\forall x \ \exists y \ P(x, y)$</td>
<td>$(x_1, y_1), (x_2, y_2), (x_3, y_3)$</td>
<td></td>
</tr>
<tr>
<td>$\exists y \ \forall x \ P(x, y)$</td>
<td>We can find ONE $y$ that works no matter what $x$ is.</td>
<td>For any candidate $y$, there is an $x$ that it doesn’t work for.</td>
</tr>
<tr>
<td>$\exists y \ \forall x \ P(x, y)$</td>
<td>$(x_1, y), (x_2, y), (x_3, y)$</td>
<td></td>
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Lecture 6 Activity

• You will be assigned to breakout rooms. Please:
  • Introduce yourself
  • Choose someone to share screen, showing this PDF
  • Today’s task: Consider the predicate logic expression
    \[ \neg \exists x \left( (\forall y P(x, y)) \lor (\exists z Q(x, z)) \right) \]
  • Obtain an equivalent logic expression where negations are directly in front of the predicates.

Then fill out the poll everywhere for Activity Credit!
Go to pollev.com/thomas311 and login with your UW identity

De Morgan Laws:

\[ \neg \forall x P(x) \equiv \exists x \neg P(x) \]
\[ \neg \exists x P(x) \equiv \forall x \neg P(x) \]
Logical Inference

• So far we’ve considered:
  – How to understand and express things using propositional and predicate logic
  – How to *compute* using Boolean (propositional) logic
  – How to show that different ways of expressing or computing them are *equivalent* to each other

• Logic also has methods that let us *infer* implied properties from ones that we know
  – Equivalence is a small part of this
Applications of Logical Inference

• **Software Engineering**
  – Express desired properties of program as set of logical constraints
  – Use inference rules to show that program implies that those constraints are satisfied

• **Artificial Intelligence**
  – Automated reasoning

• **Algorithm design and analysis**
  – e.g., Correctness, Loop invariants.

• **Logic Programming, e.g. Prolog**
  – Express desired outcome as set of constraints
  – Automatically apply logic inference to derive solution
Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set
An inference rule: *Modus Ponens*

- If \( p \) and \( p \rightarrow q \) are both true then \( q \) must be true

- Write this rule as \( p, p \rightarrow q \)
  \[
  \therefore q
  \]

- Given:
  - If it is Friday, then you have a 311 class today.
  - It is Friday.

- Therefore, by Modus Ponens:
  - You have a 311 class today.
Inference Rules

If $A$ is true and $B$ is true ....

Requirements: $A \ ; \ B$
Conclusions: $\therefore C , D$

Then, $C$ must be true
Then $D$ must be true

Example (Modus Ponens):

$A \ ; \ A \rightarrow B$

$\therefore B$

If I have $A$ and $A \rightarrow B$ both true, Then $B$ must be true.
Axioms: Special inference rules

If I have nothing...

Requirements:  

Conclusions:  \[ \therefore C, D \]

Then, \( C \) must be true

Then, \( D \) must be true

Example (Excluded Middle):

\[ \therefore A \lor \lnot A \]

\( A \lor \lnot A \) must be true.
My First Proof!

Show that $s$ follows from $q$, $q \rightarrow r$, and $r \rightarrow s$

1. $q$ Given
2. $q \rightarrow r$ Given
3. $r \rightarrow s$ Given
4. 
5. 
Show that $s$ follows from $q$, $q \rightarrow r$, and $r \rightarrow s$

1. $q$         Given
2. $q \rightarrow r$   Given
3. $r \rightarrow s$   Given
4. $r$         MP: 1, 2
5. $s$         MP: 3, 4
Proofs can use equivalences too

Show that $\neg q$ follows from $q \rightarrow r$ and $\neg r$

1. $q \rightarrow r$  Given
2. $\neg r$  Given
3. 
4. 
Proofs can use equivalences too

Show that \( \neg q \) follows from \( q \rightarrow r \) and \( \neg r \)

1. \( q \rightarrow r \)  \hspace{20pt} \text{Given}
2. \( \neg r \)  \hspace{20pt} \text{Given}
3. \( \neg r \rightarrow \neg q \)  \hspace{20pt} \text{Contrapositive: 1}
4. 

Show that \( \neg q \) follows from \( q \rightarrow r \) and \( \neg r \)

1. \( q \rightarrow r \)  
   Given
2. \( \neg r \)  
   Given
3. \( \neg r \rightarrow \neg q \)  
   Contrapositive: 1
4. \( \neg q \)  
   MP: 2, 3
Simple Propositional Inference Rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

\[
\frac{q \land r}{\therefore q, r}
\]

\[
\frac{q \lor r, \neg q}{\therefore r}
\]

\[
\frac{q, r}{\therefore q \land r}
\]

\[
\frac{q}{\therefore q \lor r}
\]

\[
\frac{q, q \rightarrow r}{\therefore r}
\]

\[
\frac{p \Rightarrow q}{\therefore p \rightarrow q}
\]

Direct Proof Rule

Not like other rules
Proofs

Show that \( r \) follows from \( p, p \rightarrow q \) and \( (p \land q) \rightarrow r \)

How To Start:
We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

\[
p, p \rightarrow q
\]
\[
\therefore q
\]

\[
p \land q
\]
\[
\therefore p, q
\]

\[
p, q
\]
\[
\therefore p \land q
\]
Proofs

Show that $r$ follows from $p, p \rightarrow q$, and $p \land q \rightarrow r$

1. $p$  
   Given
2. $p \rightarrow q$  
   Given
3.
4.
5.
6.
Show that $r$ follows from $p, p \rightarrow q$, and $p \land q \rightarrow r$

1. $p$ Given
2. $p \rightarrow q$ Given
3. $q$ MP: 1, 2
4. 
5. 
6.
Show that \( r \) follows from \( p, p \to q, \) and \( p \land q \to r \)

1. \( p \)  
   Given
2. \( p \to q \)  
   Given
3. \( q \)  
   MP: 1, 2
4. \( p \land q \)  
   Intro \( \land: 1, 3 \)
Show that \( r \) follows from \( p, p \rightarrow q, \) and \( p \land q \rightarrow r \)

1. \( p \)  
   Given
2. \( p \rightarrow q \)  
   Given
3. \( q \)  
   \( \text{MP: 1, 2} \)
4. \( p \land q \)  
   \( \text{Intro } \land: 1, 3 \)
5. \( p \land q \rightarrow r \)  
   Given
6.
Show that \( r \) follows from \( p, p \rightarrow q, \) and \( p \land q \rightarrow r \)

1. \( p \)  
   Given
2. \( p \rightarrow q \)  
   Given
3. \( q \)  
   MP: 1, 2
4. \( p \land q \)  
   Intro \( \land: \) 1, 3
5. \( p \land q \rightarrow r \)  
   Given
6. \( r \)  
   MP: 4, 5
Show that $r$ follows from $p, p \rightarrow q,$ and $p \land q \rightarrow r$

1. $p$ \hspace{1cm} Given
2. $p \rightarrow q$ \hspace{1cm} Given
3. $q$ \hspace{1cm} MP: 1, 2
4. $p \land q$ \hspace{1cm} Intro $\land$: 1, 3
5. $p \land q \rightarrow r$ \hspace{1cm} Given
6. $r$ \hspace{1cm} MP: 4, 5

Two visuals of the same proof.
We will use the top one, but if the bottom one helps you think about it, that’s great!
Important: Applications of Inference Rules

• You can use equivalences to make substitutions of any sub-formula.

• Inference rules only can be applied to whole formulas (not correct otherwise).

  e.g. 1. \( p \rightarrow q \) given

  2. \( (p \lor r) \rightarrow q \) intro \( \lor \) from 1.

Does not follow!  e.g. \( p=F, q=F, r=T \)
Prove that \( \neg r \) follows from \( p \land s \), \( q \to \neg r \), and \( \neg s \lor q \).

First: Write down givens and goal.

1. \( p \land s \) Given
2. \( q \to \neg r \) Given
3. \( \neg s \lor q \) Given

Idea: Work backwards!
Proofs

Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

Idea: Work backwards!

We want to eventually get \( \neg r \). How?
- We can use \( q \rightarrow \neg r \) to get there.
- The justification between 2 and 20 looks like “elim \( \rightarrow \)” which is MP.

20. \( \neg r \) MP: 2, ?
Prove that $\neg r$ follows from $p \land s$, $q \rightarrow \neg r$, and $\neg s \lor q$.

1. $p \land s$  Given
2. $q \rightarrow \neg r$  Given
3. $\neg s \lor q$  Given

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new “hole”
- We need to prove $q$...
  - Notice that at this point, if we prove $q$, we’ve proven $\neg r$...

19. $q$
20. $\neg r$  MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s, q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \)     Given
2. \( q \rightarrow \neg r \)   Given
3. \( \neg s \lor q \)   Given

19. \( q \)     

This looks like or-elimination.

20. \( \neg r \)     MP: 2, 19
Prove that $\neg r$ follows from $p \land s$, $q \rightarrow \neg r$, and $\neg s \lor q$.

1. $p \land s$  Given
2. $q \rightarrow \neg r$  Given
3. $\neg s \lor q$  Given

18. $\neg \neg s$  \textcolor{red}{?}  
19. $q$  $\lor$ Elim: 3, 18
20. $\neg r$  MP: 2, 19

$\neg \neg s$ doesn’t show up in the givens but $s$ does and we can use equivalences
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \)  
   Given

2. \( q \rightarrow \neg r \)  
   Given

3. \( \neg s \lor q \)  
   Given

17. \( s \)  

18. \( \neg \neg s \)  
   Double Negation: 17

19. \( q \)  
   \lor Elim: 3, 18

20. \( \neg r \)  
   MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

17. \( s \) \( \land \) Elim: 1
18. \( \neg \neg s \) Double Negation: 17
19. \( q \) \( \lor \) Elim: 3, 18
20. \( \neg r \) MP: 2, 19

No holes left! We just need to clean up a bit.
Prove that \( \neg r \) follows from \( p \land s \), \( q \to \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) \hspace{1cm} \text{Given}
2. \( q \to \neg r \) \hspace{1cm} \text{Given}
3. \( \neg s \lor q \) \hspace{1cm} \text{Given}
4. \( s \) \hspace{1cm} \land \text{Elim: 1}
5. \( \neg \neg s \) \hspace{1cm} \text{Double Negation: 4}
6. \( q \) \hspace{1cm} \lor \text{Elim: 3, 5}
7. \( \neg r \) \hspace{1cm} \text{MP: 2, 6}