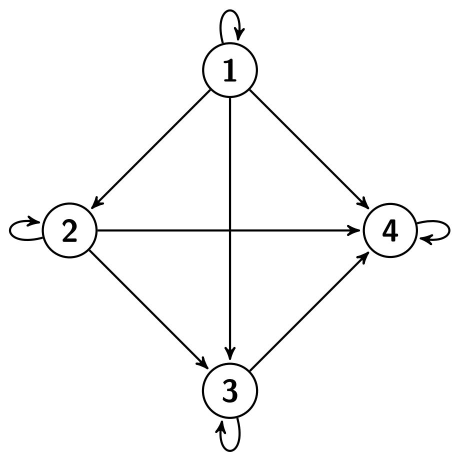
Section 08: Solutions

1. Relations

(a) Draw the transitive-reflexive closure of $\{(1,2),(2,3),(3,4)\}$.
Solution:



(b) Suppose that R is reflexive. Prove that $R \subseteq R^2$.
Solution:

Suppose $(a,b) \in R$. Since R is reflexive, we know $(b,b) \in R$ as well. Since there is a b such that $(a,b) \in R$ and $(b,b) \in R$, it follows that $(a,b) \in R^2$. Thus, $R \subseteq R^2$.

(c) Consider the relation $R = \{(x,y) : x = y+1\}$ on $\mathbb{N}$. Is R reflexive? Transitive? Symmetric? Anti-symmetric?
Solution:

It isn't reflexive, because $1 \neq 1+1$; so, $(1,1) \notin R$. It isn't symmetric, because $(2,1) \in R$ (because $2 = 1+1$), but $(1,2) \notin R$, because $1 \neq 2+1$. It isn't transitive, because note that $(3,2) \in R$ and $(2,1) \in R$, but $(3,1) \notin R$. It is anti-symmetric, because consider $(x,y) \in R$ such that $x \neq y$. Then, $x = y+1$ by definition of R. However, $(y,x) \notin R$, because $y = x-1 \neq x+1$.

(d) Consider the relation $S = \{(x,y) : x^2 = y^2\}$ on $\mathbb{R}$. Prove that S is reflexive, transitive, and symmetric.
Solution:

Consider $x \in \mathbb{R}$. Note that by definition of equality, $x^2 = x^2$; so, $(x,x) \in S$; so, S is reflexive.

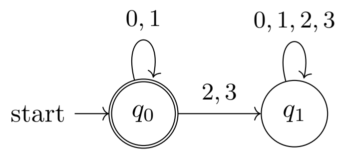
Consider $(x,y) \in S$. Then, $x^2 = y^2$. It follows that $y^2 = x^2$; so, $(y,x) \in S$. So, S is symmetric.

Suppose $(x,y) \in S$ and $(y,z) \in S$. Then, $x^2 = y^2$, and $y^2 = z^2$. Since equality is transitive, $x^2 = z^2$. So, $(x,z) \in S$. So, S is transitive.

2. DFAs, Stage 1

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1, 2, 3\}$.

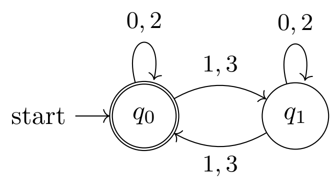
(a) All binary strings.
Solution:



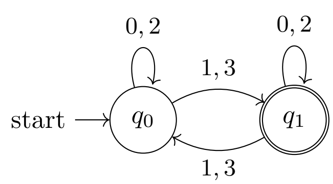
q_0**:** binary strings

q_1**:** strings that contain a character which is not 0 or 1.

(b) All strings whose digits sum to an even number.
Solution:



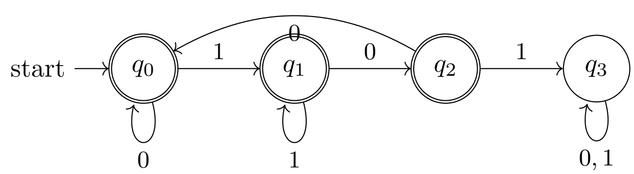
(c) All strings whose digits sum to an odd number.
Solution:



3. DFAs, Stage 2

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) All strings which do not contain the substring 101.
Solution:



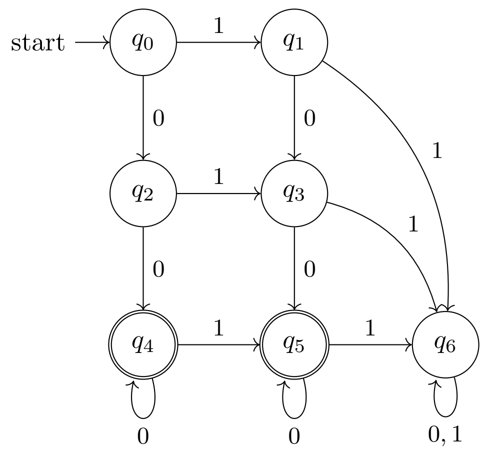
q_3**:** string that contain 101.

q_2**:** strings that don't contain 101 and end in 10.

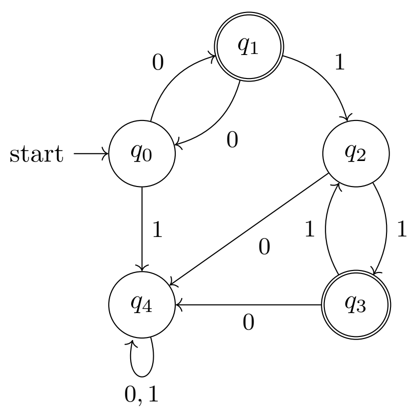
q_1**:** strings that don't contain 101 and end in 1.

q_0**:** ε, 0, strings that don't contain 101 and end in 00.

(b) All strings containing at least two 0's and at most one 1.
Solution:



(c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.
Solution:



4. Relations and Strings

Let $\Sigma = \{0, 1\}$ and define the relation $\diamond$ on Σ^* by $x \diamond y$ if and only if the length of xy is even. (Here $x \diamond y$ is another way of writing $(x, y) \in \diamond$.) Prove that $\diamond$ is reflexive, symmetric, and transitive.

Solution:

Reflexivity: Consider $a \in \Sigma^*$. Case 1: The length of a is odd. Length(aa) = even. Case 2: The length of a is even. Length(aa) = even. So, $a \diamond a$ and $\diamond$ is reflexive.

Symmetric: Suppose $a \diamond b$. Then, the length of ab is even. Length of ba is the same as length ab. So, the length of ba is even. So, $b \diamond a$ and $\diamond$ is symmetric.

Transitivity: Suppose $a \diamond b$ and $b \diamond c$. Then, the length of ab is even and the length of bc is even.

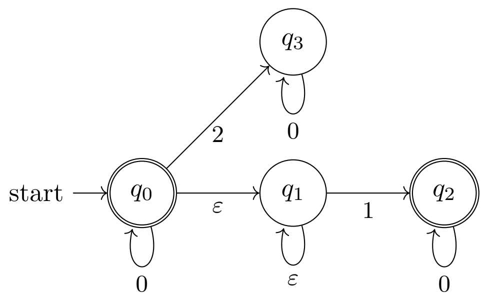
Case 1: The length of a and b are even. Then, the length of c must also be even, since bc has even length. Then, a and c have even length, so ac has even length.

Case 2: The length of a and b are odd. Then, the length of c must also be odd, since bc has even length. Then, a and c have odd length, so ac has even length.

So, $a \diamond c$ and $\diamond$ is transitive.

5. NFAs

(a) What language does the following NFA accept?

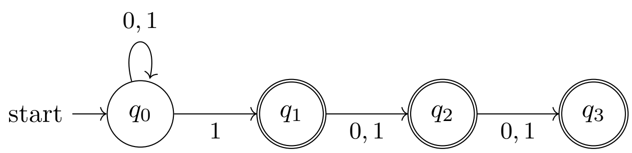


Solution:

All strings of only 0's and 1's not containing more than one 1.

(b) Create an NFA for the language "all binary strings that have a 1 as one of the last three digits".
Solution:

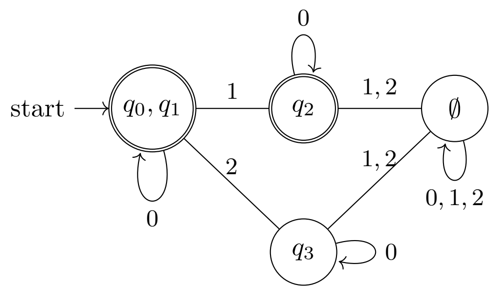
The following is one such NFA:



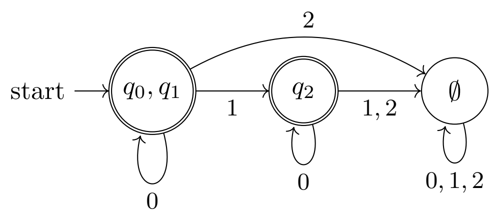
6. DFAs & Minimization

(a) Convert the NFA from 1a to a DFA, then minimize it.

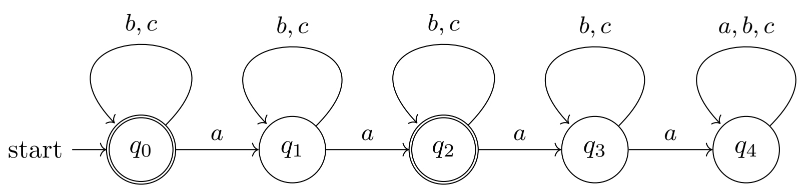
Solution:



Here is the minimized form:



(b) Minimize the following DFA:



Solution:

Step 1: q_0, q_2 are final states and the rest are not final. So, we start with the initial partition with the following groups: group 1 is $\{q_0, q_2\}$ and group 2 is $\{q_1, q_3, q_4\}$.

Step 2: q_1 is sending a to group 1 while q_3, q_4 are sending a to group 2. So, we divide group 2. We get the following groups: group 1 is $\{q_0, q_2\}$, group 3 is $\{q_1\}$ and group 4 is $\{q_3, q_4\}$.

Step 3: q_0 is sending a to group 3 and q_2 is sending a to group 4. So, we divide group 1. We will have the following groups: group 3 is $\{q_1\}$, group 4 is $\{q_3, q_4\}$, group 5 is $\{q_0\}$ and group 6 is $\{q_2\}$.

The minimized DFA is the following: