

Section 06: Induction

1. GCD

- (a) Calculate $\gcd(100, 50)$.
- (b) Calculate $\gcd(17, 31)$.
- (c) Find the multiplicative inverse of $6 \pmod{7}$.
- (d) Does 49 have an multiplicative inverse $\pmod{7}$?

2. Extended Euclidean Algorithm

- (a) Find the multiplicative inverse y of $7 \pmod{33}$. That is, find y such that $7y \equiv 1 \pmod{33}$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leq y < 33$.
- (b) Now, solve $7z \equiv 2 \pmod{33}$ for all of its integer solutions z .

3. A Horse of a Different Color

Did you know that all dogs are named Dubs? It's true. Maybe. Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n P(n)$ by induction on n .

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k .

Inductive Step: Consider an arbitrary group of $k + 1$ dogs. Arbitrarily select a dog, D , and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D' . We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k + 1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k + 1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

4. Induction with Equality

- (a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.
- (b) Define the triangle numbers as $\Delta_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\Delta_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \Delta_n^2$$

5. Induction with Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

6. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

7. Induction with Formulas

These problems are a little more difficult and abstract. Try making sure you can do all the other problems before trying these ones.

- (a) (i) Show that given two sets A and B that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)
- (ii) Show using induction that for an integer $n \geq 2$, given n sets $A_1, A_2, \dots, A_{n-1}, A_n$ that
- $$\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$$
- (b) (i) Show that given any integers a, b , and c , if $c \mid a$ and $c \mid b$, then $c \mid (a + b)$. (Don't use induction.)
- (ii) Show using induction that for any integer $n \geq 2$, given n numbers $a_1, a_2, \dots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \dots, n$, that

$$c \mid (a_1 + a_2 + \cdots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

8. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in year n is described by the function $f(n)$:

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(n) &= 2f(n-1) - f(n-2) \text{ for } n \geq 2 \end{aligned}$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n . That is, construct a formula for $f(n)$ and prove its correctness.