Section 04: Solutions

1. Formal Spoofs

For each of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $\exists z\, \forall x\, P(x, z)$ follows from $\forall x\, \exists y\, P(x, y)$.

1.	$\forall x\, \exists y\, P(x, y)$	Given
2.	$\forall x\, P(x, c)$	$\exists$ Elim: 1 (c special)
3.	$\exists z\, \forall x P(x, z)$	$\exists$ Intro: 2

Solution:

The mistake is on line 2 where an inference rule is used on a subexpression. When we apply something like the $\exists$ Elim rule, the $\exists$ must be at the start of the expression and outside all other parts of the statement.

The conclusion is false, it's basically saying we can interchange the order of $\forall$ and $\exists$ quantifiers. Let the domain of discourse be integers and define $P(x, y)$ to be $x < y$. Then the hypothesis is true: for every integer, there is a larger integer. However, the conclusion is false: there is no integer that is larger than every other integer. Hence, there can be no correct proof that the conclusion follows from the hypothesis.

(b) Show that $\exists z\, (P(z) \land Q(z))$ follows from $\forall x\, P(x)$ and $\exists y\, Q(y)$.

1.	$\forall x\, P(x)$	Given
2.	$\exists y\, Q(y)$	Given
3.	Let z be arbitrary	
4.	$P(z)$	Elim $\forall$: 1
5.	$Q(z)$	Elim $\exists$: 2 (z special)
6.	$P(z) \land Q(z)$	Intro $\land$: 4, 5
7.	$\exists z\, P(z) \land Q(z)$	Intro $\exists$: 6

Solution:

The mistake is on line 5. The $\exists$ Elim rule must create a new variable rather than applying some property to an existing variable.

The conclusion is true in this case. Instead of declaring z to be arbitrary and then applying $\exists$ Elim to make it specific, we can instead just apply the $\exists$ Elim rule directly to create z. To do this, we would remove lines 3 and 5 and define z by applying $\exists$ Elim to line 2. Note, it's important that we define z before applying line 4.

2. Predicate Logic Formal Proof

Given $\forall x\, T(x) \to M(x)$, we wish to prove $(\exists x\, T(x)) \to (\exists y\, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1. $\forall x\, T(x) \to M(x)$ (________________)
    - 2.1. $\exists x\, T(x)$ (________________)
    - 2.2. $T(c)$ (________________)
    - 2.3. $T(c) \to M(c)$ (________________)
    - 2.4. $M(c)$ (________________)
    - 2.5. $\exists y\, M(y)$ (________________)
2. $(\exists x\, T(x)) \to (\exists y\, M(y))$ (________________)

Solution:

1.	$\forall x\, T(x) \to M(x)$	Given
2.1.	$\exists x.\, T(x)$	Assumption
2.2.	$T(c)$	Elim ∃: 2.1 (c)
2.3.	$T(c) \to M(c)$	Elim ∀: 1
2.4.	$M(c)$	Modus Ponens: 2.2, 2.3
2.5.	$\exists y\, M(y)$	Intro ∃: 2.4
2.	$(\exists x\, T(x)) \to (\exists y\, M(y))$	Direct Proof: 2.1-2.5

3. A Formal Proof in Predicate Logic

Prove $\exists x\ (P(x) \lor R(x))$ from $\forall x\ (P(x) \lor Q(x))$ and $\forall y\ (\neg Q(y) \lor R(y))$.

Solution:

1.	$\forall x\ (P(x) \lor Q(x))$	Given
2.	$\forall y\ (\neg Q(y) \lor R(y))$	Given
3.	$P(a) \lor Q(a)$	Elim ∀: 1
4.	$\neg Q(a) \lor R(a)$	Elim ∀: 2
5.	$Q(a) \to R(a)$	Law of Implication: 4
6.	$\neg\neg P(a) \lor Q(a)$	Double Negation: 3
7.	$\neg P(a) \to Q(a)$	Law of Implication: 5
8.1.	$\neg P(a)$	Assumption
8.2.	$Q(a)$	Modus Ponens: 8.1, 7
8.3.	$R(a)$	Modus Ponens: 8.2, 5
8.	$\neg P(a) \to R(a)$	Direct Proof
9.	$\neg\neg P(a) \lor R(a)$	Law of Implication: 8
10.	$P(a) \lor R(a)$	Double Negation: 9
11.	$\exists x\ (P(x) \lor R(x))$	Intro ∃: 10

4. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

Solution:

3

(b) $B = \{\{\},\ \{\{\}\},\ \{\{\},\{\}\},\ \{\{\},\{\},\{\}\},\ \dots\}$

Solution:

$$\begin{aligned} B &= \{\{\},\ \{\{\}\},\ \{\{\},\{\}\},\ \{\{\},\{\},\{\}\},\ \dots\} \\ &= \{\{\},\ \{\{\}\},\ \{\{\}\},\ \{\{\}\},\ \dots\} \\ &= \{\varnothing, \{\varnothing\}\} \end{aligned}$$

So, there are two elements in B.

(c) $D = \varnothing$

Solution:

0.

(d) $E = \{\varnothing\}$

Solution:

1.

(e) $C = A \times (B \cup \{7\})$

Solution:

$C = \{1, 2, 3\} \times \{\varnothing, \{\varnothing\}, 7\} = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \{\varnothing, \{\varnothing\}, 7\}\}$. It follows that there are $3 \times 3 = 9$ elements in C.

5. Game, Set, Match

Prove each of the following set identities.

(a) $A \setminus B \subseteq A \cup C$ for any sets A, B, C. (Formally and then in English.)

Solution:

1. Let x be an arbitrary object.

2.1.	$x \in A \setminus B$	Assumption
2.2.	$(x \in A) \wedge \neg(x \in B)$	Def of "$\setminus$": 2.1
2.3.	$x \in A$	Elim $\wedge$: 2.2
2.4.	$(x \in A) \vee (x \in C)$	Intro $\vee$: 2.3
2.5.	$x \in A \cup C$	Def of $\cup$: 2.4

2.	$(x \in A \setminus B) \rightarrow (x \in A \cup C)$	Direct Proof
3.	$\forall x\, ((x \in A \setminus B) \rightarrow (x \in A \cup C))$	Intro $\forall$: 1, 2
4.	$A \setminus B \subseteq A \cup C$	Def of $\subseteq$: 3

Let x be an arbitrary object. Suppose that $x \in A \setminus B$. By definition, this means that $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup C$ by the definition of $\cup$. Since x was arbitrary, this shows $A \setminus B \subseteq A \cup C$.

(b) $(A \setminus B) \setminus C \subseteq A \setminus C$ for any sets A, B. (Formally and then in English)

Solution:

1. Let x be arbitrary.

2.1.	$x \in (A \setminus B) \setminus C$	Assumption
2.2.	$(x \in A \setminus B) \wedge (x \notin C)$	Def of "$\setminus$": 2.1
2.3.	$(x \in A) \wedge (x \notin B)$	Def of "$\setminus$": 2.2
2.4.	$x \in A$	Elim $\wedge$: 2.3
2.5.	$x \notin C$	Elim $\wedge$: 2.2
2.6.	$(x \in A) \wedge (x \notin C)$	Intro $\wedge$: 2.4, 2.5
2.7.	$x \in A \setminus C$	Def of "$\setminus$": 2.6

2.	$(x \in (A \setminus B) \setminus C) \rightarrow (x \in A \setminus C)$	Direct Proof
3.	$\forall x\, ((x \in (A \setminus B) \setminus C) \rightarrow (x \in A \setminus C)$	Intro $\forall$: 1, 2
4.	$(A \setminus B) \setminus C \subseteq A \setminus C$	Def of $\subseteq$: 3

Let x be an arbitrary object. Suppose that $x \in (A \setminus B) \setminus C$. By definition, this means that $x \in A \setminus B$ and $x \notin C$ and then that $x \in A$ and $x \notin B$. The facts that $x \in A$ and $x \notin C$ show that $x \in A \setminus C$ by definition. Since x was arbitrary, this shows $(A \setminus B) \setminus C \subseteq A \setminus C$.

(c) $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D. (English only.)

Solution:

Let x be an arbitrary element of $(A \cap B) \times C$. Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$. Since $y \in A \cap B$, by definition of $\cap$, $y \in A$ (and $y \in B$). Since $z \in C$, by definition of $\cup$, we also have $z \in C \cup D$. Thus, since $y \in A$ and $z \in C \cup D$, by definition of Cartesian product we have $x = (y, z) \in A \times (C \cup D)$. Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

6. Ghosts and Skeletons

Let A and B be sets and P and Q be predicates. For each of the claims below, write the *skeleton* of an English proof of the claim. It will not be possible to complete the proof with just the information given, but you should be able to see the basic shape of the proof.

For example, suppose we want to prove "No element of A satisfies P." Then, our proof would have this shape:

> Let x be arbitrary.
> Suppose that $x \in A$. **....** Thus, $P(x)$ is false.
> Since x was arbitrary, this shows that no element of A satisfies P.

This shows the general shape (skeleton) of the proof. We don't know how to complete the proof since we don't know what A and P are. For any particular choice of A and P, though, the proof would still look like this but with the "...." replaced by specific reasoning for that A and P.

Note that we have actually proven $\forall x \, \neg P(x)$, whereas the claim best translates as $\neg \exists x \, P(x)$. However, the two are equivalent by De Morgan's law, and that is a simple enough step that the reader should see it.

(a) $A = B$

Solution:

Let x be arbitrary.

Suppose that $x \in A$. **....** Thus, $x \in B$.

Now, suppose that $x \in B$. **....** Thus, $x \in A$.

We have shown that $x \in A$ iff $x \in B$. Since x was arbitrary, the sets are equal by definition.

(b) Any object that satisfies P but not Q is in the set B.

Solution:

Let x be arbitrary.

Suppose that x satisfies P but not Q. **....** Thus, $x \in B$.

Since x was arbitrary, this shows that anything satisfying P but not Q is in B.

(c) B is not a subset of A.

Solution:

[We need to show $\neg \forall x \, ((x \in B) \rightarrow (x \in A))$, but that is equivalent to $\exists x \, ((x \in B) \wedge (x \notin A))$.]

Let $x = \ldots$. Since ..., we can see that $x \in B$. On the other hand, since ..., we can see that $x \notin A$. Thus, x is a counterexample to the claim that B is a subset of A.