1. Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) \( \neg p \rightarrow (s \rightarrow r) \) vs. \( s \rightarrow (p \lor r) \)

(b) \( p \leftrightarrow \neg p \) vs. \( F \) (Hint: recall the Biconditional rule \( p \leftrightarrow r \equiv (p \rightarrow r) \land (r \rightarrow p) \))

2. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) \( p \rightarrow r \) vs. \( r \rightarrow p \)

(b) \( a \rightarrow (b \land c) \) vs. \( (a \rightarrow b) \land c \)

3. They mean the same thing

Prove the following claims using chains of elementary equivalences, as shown in lecture:

\[
\neg (\neg q \lor r) \equiv \neg (\neg q) \land \neg r \quad (1)
\]

\[
\neg (\neg q) \land \neg r \equiv q \land \neg r \quad (2)
\]

\[
q \land \neg r \equiv \neg r \land q \quad (3)
\]

Your friend says this means that \( \neg (q \rightarrow r) \equiv \neg r \land q \). Is that true?

4. Equivalent Translations

Prove that the following English statements are equivalent.

(i) Unless it isn’t raining or I don’t have an umbrella, I buy a book.
(ii) It isn’t raining or I don’t have an umbrella or I buy a book.

5. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) \( \neg p \lor (\neg q \lor (p \land q)) \)

(b) \( \neg (p \lor (q \land p)) \)
6. Properties of XOR

Like $\land$ and $\lor$, the $\oplus$ operator (exclusive or) has many interesting properties. For example, it is easy to verify with a truth table that $\oplus$ is also associative. In this problem, we will prove some additional properties of $\oplus$.

For this problem only, you may also use the equivalence

$$p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$$

which you may cite as “Definition of $\oplus$”. This equivalence allows you to translate $\oplus$ into an expression involving only $\land$, $\lor$, and $\neg$, so that the standard equivalences can then be applied.

Prove the following claims using chains of elementary equivalences, as shown in lecture:

(a) $p \oplus q \equiv q \oplus p$ (Commutativity)

(b) $p \oplus p \equiv F$ and $p \oplus \neg p \equiv T$

(c) $p \oplus F \equiv p$ and $p \oplus T \equiv \neg p$

(d) $(\neg p) \oplus q \equiv \neg(p \oplus q) \equiv p \oplus (\neg q)$. I.e., negating one of the inputs negates the overall expression.