1. Equivalences

Prove that each of the following pairs of propositional formulas are equivalent using propositional equivalences.

(a) \( \neg p \to (s \to r) \) vs. \( s \to (p \lor r) \) **Solution:**

\[
\begin{align*}
\neg p \to (s \to r) & \equiv \neg p \lor (s \to r) \quad \text{Law of Implication} \\
& \equiv p \lor (s \to r) \quad \text{Double Negation} \\
& \equiv p \lor (\neg s \lor r) \quad \text{Law of Implication} \\
& \equiv (p \lor \neg s) \lor r \quad \text{Associativity} \\
& \equiv (\neg s \lor p) \lor r \quad \text{Commutativity} \\
& \equiv \neg s \lor (p \lor r) \quad \text{Associativity} \\
& \equiv s \to (p \lor r) \quad \text{Law of Implication}
\end{align*}
\]

(b) \( p \leftrightarrow \neg p \) vs. \( F \) **Solution:** (Hint: recall the Biconditional rule \( p \leftrightarrow r \equiv (p \to r) \land (r \to p) \))

\[
\begin{align*}
p \leftrightarrow \neg p & \equiv (p \to \neg p) \land (\neg p \to p) \quad \text{Biconditional} \\
& \equiv (\neg p \lor \neg p) \land (\neg p \lor p) \quad \text{Law of Implication} \\
& \equiv (\neg p \lor \neg p) \land (p \lor p) \quad \text{Double Negation} \\
& \equiv \neg p \land p \quad \text{Idempotence} \\
& \equiv F \quad \text{Negation}
\end{align*}
\]

2. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) \( p \to r \) vs. \( r \to p \) **Solution:**

When \( p = T \) and \( r = F \), then \( p \to r \equiv F \), but \( r \to p \equiv T \).

(b) \( a \to (b \land c) \) vs. \( (a \to b) \land c \) **Solution:**

When \( a = F \) and \( c = F \), then \( a \to (b \land c) \equiv T \) (by vacuous truth), but \( (a \to b) \land c \equiv F \) (because \( c \) is false).

3. They mean the same thing

Prove the following claims using chains of elementary equivalences, as shown in lecture:

\[
\begin{align*}
\neg (\neg q \lor r) & \equiv \neg (\neg q) \land \neg r \quad (1) \\
\neg (\neg q) \land \neg r & \equiv q \land \neg r \quad (2) \\
q \land \neg r & \equiv \neg r \land q \quad (3)
\end{align*}
\]
Your friend says this means that \( \neg(q \rightarrow r) \equiv \neg r \land q \). Is that true?

**Solution:**

\[
\begin{align*}
\neg(q \rightarrow r) & \equiv \neg(\neg q \lor r) & \text{Law of Implication} \\
& \equiv \neg q \land \neg r & \text{De Morgan} \\
& \equiv q \land \neg r & \text{Double Negation} \\
& \equiv \neg r \land q & \text{Commutativity}
\end{align*}
\]

For any statements \( A, B, \) and \( C \), if \( A \) and \( B \) agree on all possible truth assignments and \( B \) and \( C \) do too, then \( A \) and \( C \) agree on all possible truth assignments, so the above chain of equivalences shows that \( \neg(q \rightarrow r) \equiv \neg r \land q \).

### 4. Equivalent Translations

Prove that the following English statements are equivalent.

(i) Unless it isn’t raining or I don’t have an umbrella, I buy a book.

(ii) It isn’t raining or I don’t have an umbrella or I buy a book.  **Solution:**

\[
\begin{align*}
a & : \text{It is raining.} \\
b & : \text{I have an umbrella.} \\
c & : \text{I buy a book.}
\end{align*}
\]

When we say unless \( a, b \), this suggests that as long as \( a \) is not true, \( b \) will be true. Then, we can rewrite (i) as follows:

\[
\neg(\neg a \lor \neg b) \rightarrow c
\]

With the same propositional variables, we can rewrite (ii) as:

\[
\neg a \lor \neg b \lor c
\]

If these two compound propositions are equivalent, then the English statements are equivalent. Starting with the left-hand side

\[
\neg(\neg a \lor \neg b) \rightarrow c \equiv (\neg a \land \neg b) \rightarrow c
\]

\[
\equiv (a \land b) \rightarrow c & \text{ De Morgan} \\
\equiv \neg(a \land b) \rightarrow c & \text{ Double Negation} \\
\equiv \neg(a \land b) \lor c & \text{ Law of Implication} \\
\equiv (\neg a \lor \neg b) \lor c & \text{ De Morgan}
\]

Therefore, we’ve shown that the two English statements are equivalent.

### 5. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) \( \neg p \lor (\neg q \lor (p \land q)) \)  **Solution:**

First, we replace \( \neg, \lor, \) and \( \land \). This gives us \( p' + q' + pq \); note that the parentheses are not necessary in boolean algebra, because the operations are all commutative and associative. We can use DeMorgan’s laws.
to get the slightly simpler $(pq)' + pq$. Then, we can use commutativity to get $pq + (pq)'$ and complementarity to get 1. (Note that this is another way of saying the formula is a tautology.)

(b) $\neg(p \lor (q \land p))$

Solution:

First, we replace $\neg, \lor,$ and $\land$ with their corresponding boolean operators, giving us $(p + (qp))'$. Applying DeMorgan’s laws once gives us $p'(qp)'$, and a second time gives us $p'(q' + p')$, which is $p'(p' + q')$ by commutativity. By absorption, this is simply $p'$.

6. Properties of XOR

Like $\land$ and $\lor$, the $\oplus$ operator (exclusive or) has many interesting properties. For example, it is easy to verify with a truth table that $\oplus$ is also associative. In this problem, we will prove some additional properties of $\oplus$.

For this problem only, you may also use the equivalence

$$p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$$

which you may cite as “Definition of $\oplus$”. This equivalence allows you to translate $\oplus$ into an expression involving only $\land, \lor, \text{and } \neg$, so that the standard equivalences can then be applied.

Prove the following claims using chains of elementary equivalences, as shown in lecture:

(a) $p \oplus q \equiv q \oplus p$ (Commutativity)

Solution:

$$
egin{align*}
p \oplus q &\equiv (p \land \neg q) \lor (\neg p \land q) & \text{Definition of } \oplus \\
&\equiv (\neg p \land q) \lor (p \land \neg q) & \text{Commutativity} \\
&\equiv (q \land \neg p) \lor (\neg q \land p) & \text{Commutativity} \\
&\equiv q \oplus p & \text{Definition of } \oplus
\end{align*}
$$
(b) $p \oplus p \equiv F$ and $p \oplus \neg p \equiv T$

Solution:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \oplus p \equiv (p \land \neg p) \lor (\neg p \land p)$</td>
<td>Definition of $\oplus$</td>
</tr>
<tr>
<td>$\equiv (p \land \neg p) \lor (p \land \neg p)$</td>
<td>Commutativity</td>
</tr>
<tr>
<td>$\equiv (p \land \neg p)$</td>
<td>Idempotency</td>
</tr>
<tr>
<td>$\equiv F$</td>
<td>Negation</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expression</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \oplus \neg p \equiv (p \land \neg \neg p) \lor (\neg p \lor \neg p)$</td>
<td>Definition of $\oplus$</td>
</tr>
<tr>
<td>$\equiv (p \land \neg p) \lor (\neg p \lor \neg p)$</td>
<td>Double Negation</td>
</tr>
<tr>
<td>$\equiv p \lor \neg p$</td>
<td>Idempotency</td>
</tr>
<tr>
<td>$\equiv T$</td>
<td>Negation</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{(c) } p \oplus F & \equiv p \text{ and } p \oplus T \equiv \neg p \\
\end{align*}
\]

Solution:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \oplus F \equiv (p \land \neg F) \lor (\neg p \land F)$</td>
<td>Definition of $\oplus$</td>
</tr>
<tr>
<td>$\equiv (p \land (\neg F \lor F)) \lor (\neg p \land F)$</td>
<td>Identity</td>
</tr>
<tr>
<td>$\equiv (p \land (F \lor \neg F)) \lor (\neg p \land F)$</td>
<td>Commutativity</td>
</tr>
<tr>
<td>$\equiv (p \land T) \lor (\neg p \land F)$</td>
<td>Negation</td>
</tr>
<tr>
<td>$\equiv p \lor (\neg p \land F)$</td>
<td>Identity</td>
</tr>
<tr>
<td>$\equiv p \lor \neg F$</td>
<td>Domination</td>
</tr>
<tr>
<td>$\equiv p$</td>
<td>Identity</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expression</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \oplus T \equiv (p \land \neg T) \lor (\neg p \land T)$</td>
<td>Definition of $\oplus$</td>
</tr>
<tr>
<td>$\equiv (p \land \neg T) \lor \neg p$</td>
<td>Identity</td>
</tr>
<tr>
<td>$\equiv (\neg p \land \neg T) \lor \neg p$</td>
<td>Double Negation</td>
</tr>
<tr>
<td>$\equiv \neg(p \lor T) \lor \neg p$</td>
<td>De Morgan</td>
</tr>
<tr>
<td>$\equiv \neg T \lor \neg p$</td>
<td>Domination</td>
</tr>
<tr>
<td>$\equiv \neg(T \land p)$</td>
<td>De Morgan</td>
</tr>
<tr>
<td>$\equiv \neg(p \land T)$</td>
<td>Commutativity</td>
</tr>
<tr>
<td>$\equiv \neg p$</td>
<td>Identity</td>
</tr>
</tbody>
</table>
(d) \((-p) \oplus q \equiv \neg(p \oplus q) \equiv p \oplus (\neg q)\). I.e., negating one of the inputs negates the overall expression.

Solution:

\[
\neg(p \oplus q) \equiv \neg((p \land \neg q) \lor (\neg p \land q))
\]

\[
\equiv \neg(p \land \neg q) \land \neg(\neg p \land q)
\]

\[
\equiv (\neg p \lor \neg q) \land (\neg \neg p \lor \neg q)
\]

\[
\equiv ((\neg p \lor q) \land p) \lor ((\neg p \lor q) \land \neg q)
\]

\[
\equiv (p \land (\neg p \lor q)) \lor (\neg q \land (\neg p \lor q))
\]

\[
\equiv ((p \land \neg p) \lor (p \land q)) \lor ((\neg q \land \neg p) \lor (q \land \neg q))
\]

\[
\equiv (\neg q \land \neg p) \lor (q \land \neg q)
\]

\[
\equiv (\neg q \land \neg p) \lor (\neg q \land \neg p)
\]

\[
\equiv (\neg q \land \neg p) \lor (\neg q \land \neg p)
\]

\[
\equiv (\neg q \land \neg p) \lor (\neg q \land \neg p)
\]

The second equivalence \(-p \oplus q \equiv p \oplus (\neg q)\) follows from the first and Commutativity (part a).