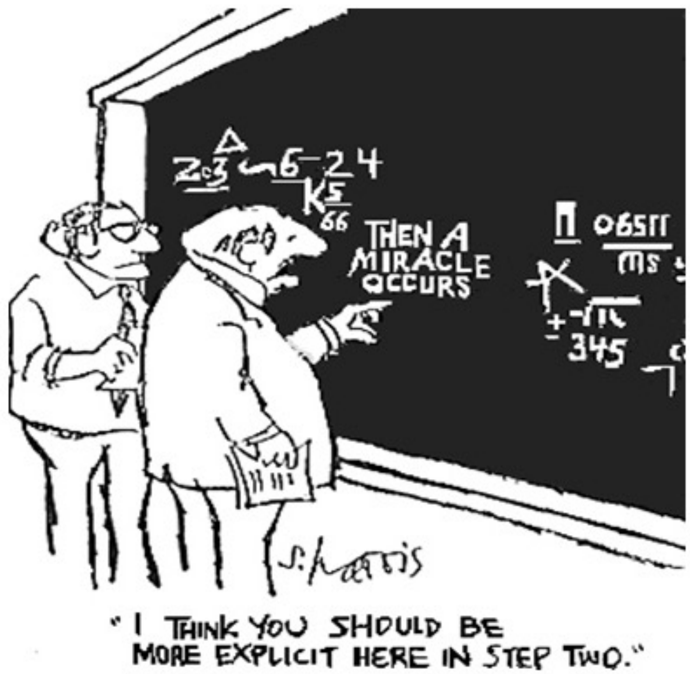
CSE 311: Foundations of Computing

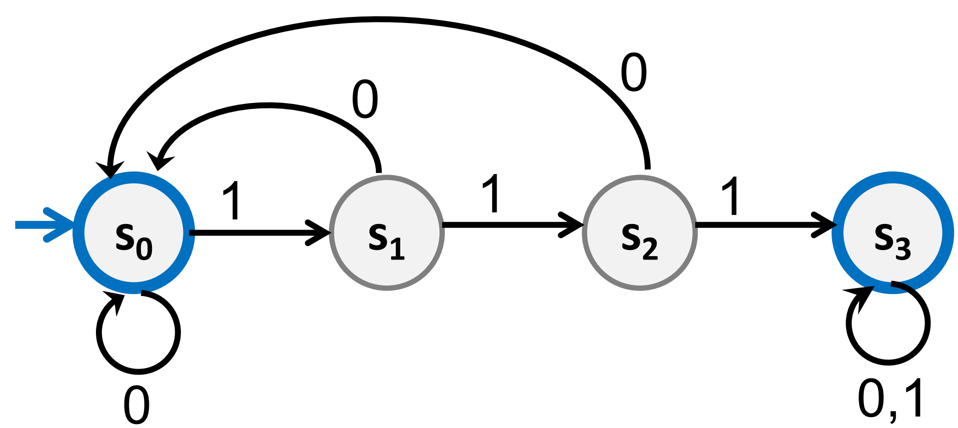
Lecture 25: NFAs and their relation to REs & DFAs



Recall: DFAs

- States
- Transitions on input symbols
- Start state and final states
- The “language recognized” by the machine is the set of strings that reach a final state from the start

Old State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3

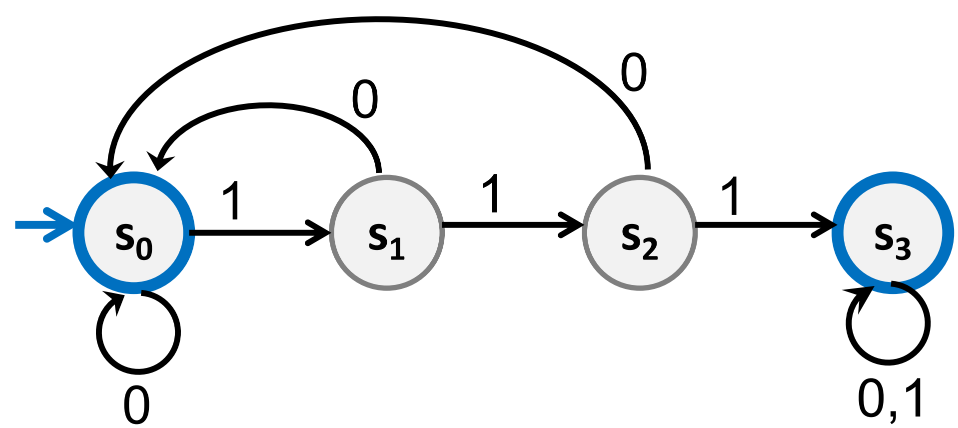


Recall: DFAs

- Each machine designed for strings over some fixed alphabet Σ.

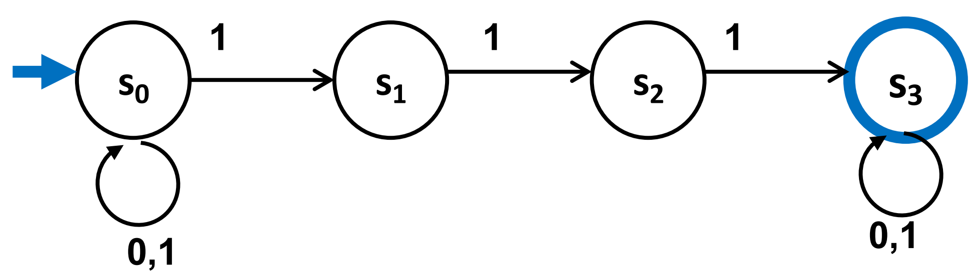
- Must have a transition defined from each state for ***every*** symbol in Σ.

Old State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3



Last Time: Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
  - Also can have edges labeled by empty string ε
- **Definition:** x is in the language recognized by an NFA if and only if <u>some</u> valid execution of the machine gets to an accept state



Three ways of thinking about NFAs

- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)

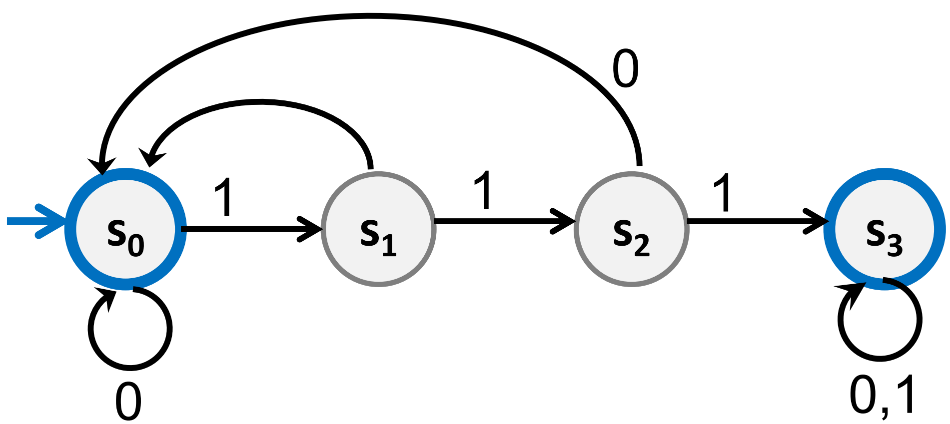
- Outside observer: Is there a path labeled by x from the start state to some accepting state?

- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

Path Labels

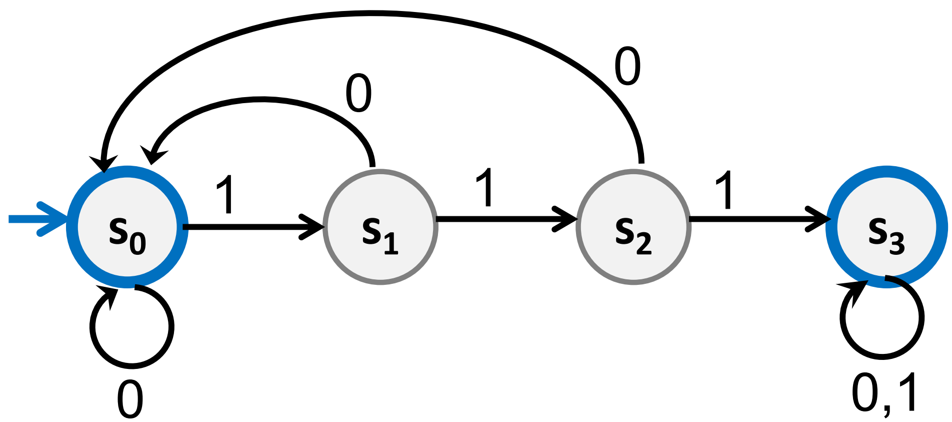
Def: The label of path $v_0, v_1, \ldots, v_n$ is the concatenation of the labels of the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$

Example: The label of path s_0, s_1, s_2, s_0, s_0 is 1100



Deterministic Finite Automata (DFA)

- **Def:** x is in the language recognized by an DFA if and only if x labels a path from the start state to some final state



- Path $v_0, v_1, \ldots, v_n$ with $v_0 = s_0$ and label x describes a correct simulation of the DFA on input x
  - i-th step must match the i-th character of x

Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
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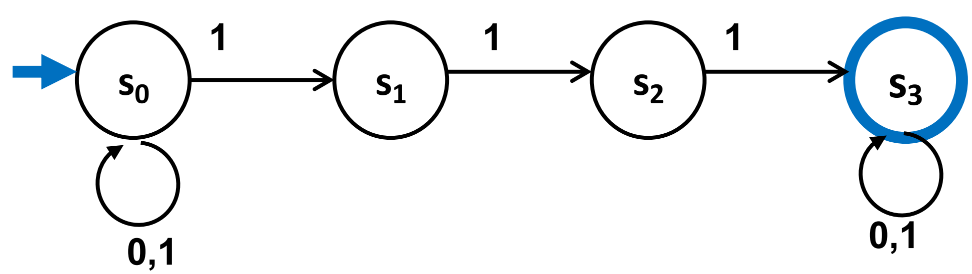

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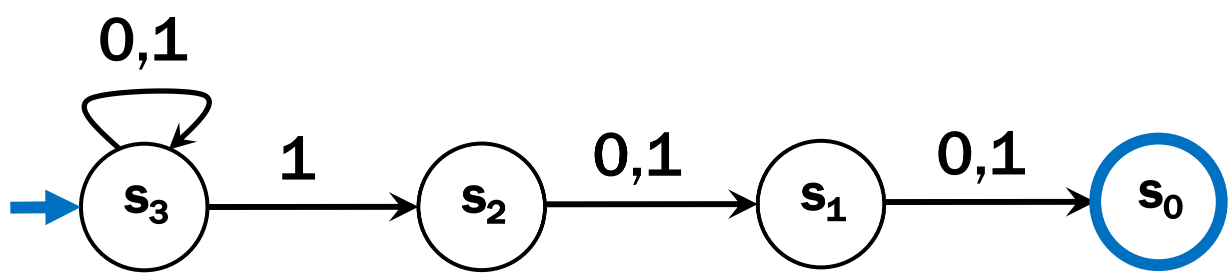
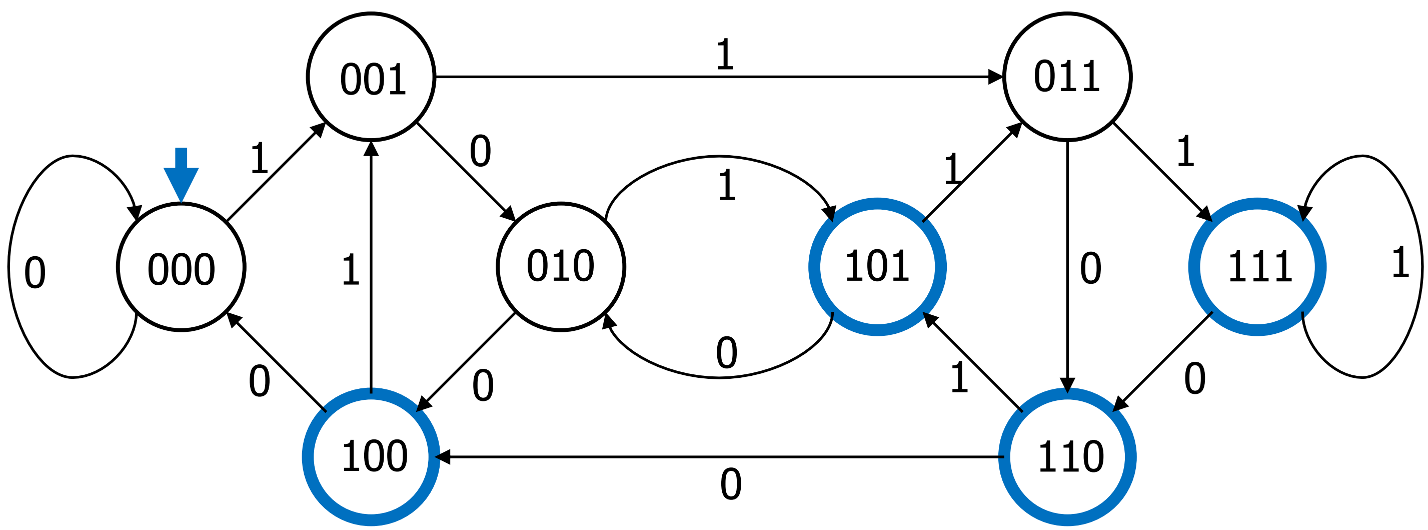
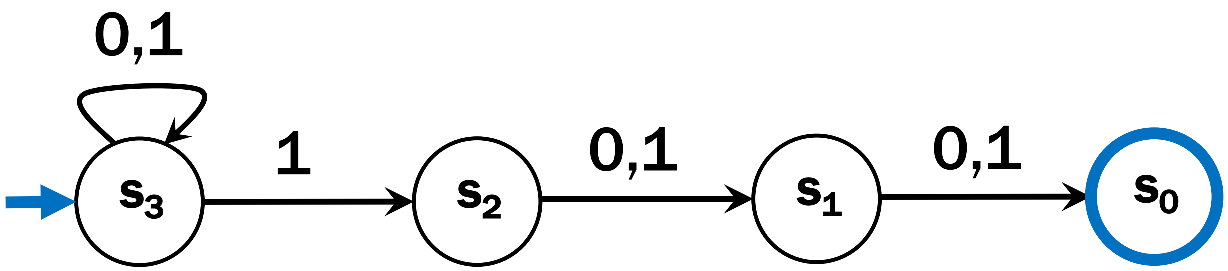
- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

Compare with the smallest DFA

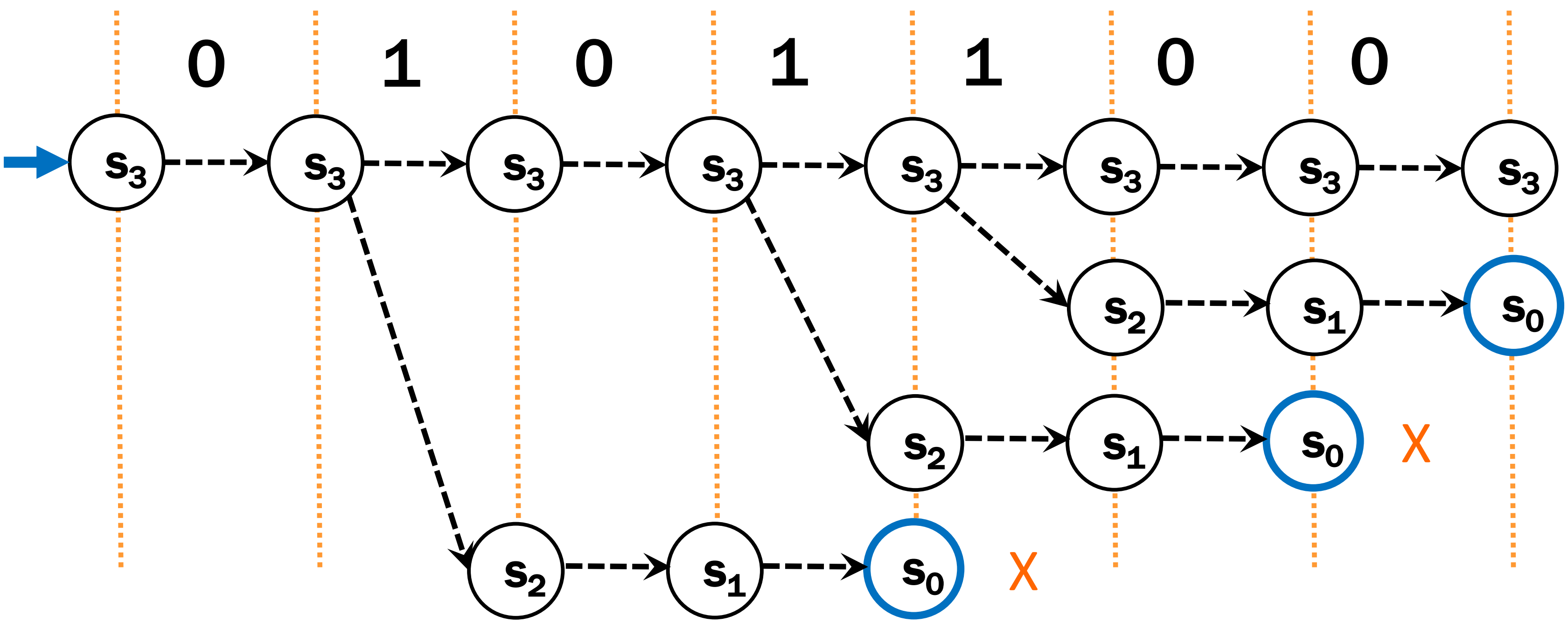




Parallel Exploration view of an NFA



Input string 0101100



Summary of NFAs

- Generalization of DFAs
  - drop two restrictions of DFAs
  - every DFA is an NFA
- *Seem* to be more powerful
  - designing is easier than with DFAs
- *Seem* related to regular expressions

The story so far...

REs	$\subseteq$	**CFGs**
DFAs	$\subseteq$	**NFAs**

NFAs and regular expressions

Theorem: For any set of strings (language) A described by a regular expression, there is an NFA that recognizes A.

Proof idea: Structural induction based on the recursive definition of regular expressions...

Regular Expressions over Σ

- **Basis:**
  - ε is a regular expression
  - ***a*** is a regular expression for any $a \in \Sigma$
- **Recursive step:**
  - If **A** and **B** are regular expressions then so are:

    A $\cup$ **B**

    AB

    A*

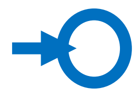
Base Case

- Case ε:

- Case *a*:

Base Case

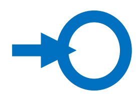
- **Case ε:**



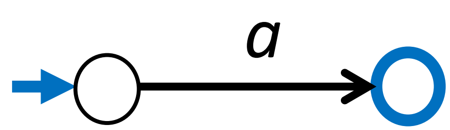
- **Case *a*:**

Base Case

- **Case ε:**

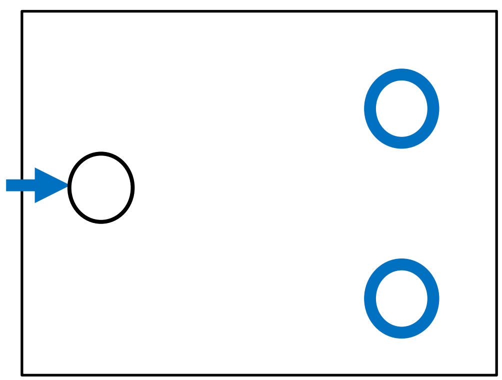


- **Case *a*:**

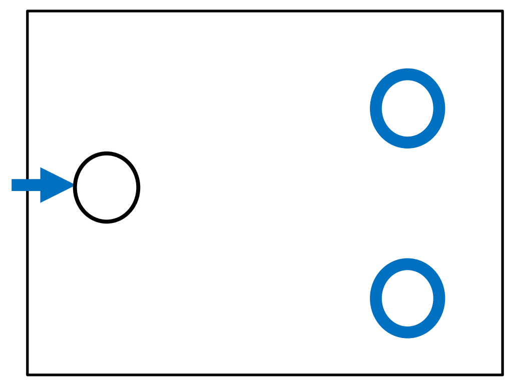


Inductive Hypothesis

- Suppose that for some regular expressions A and B there exist NFAs N_A and N_B such that N_A recognizes the language given by A and N_B recognizes the language given by B



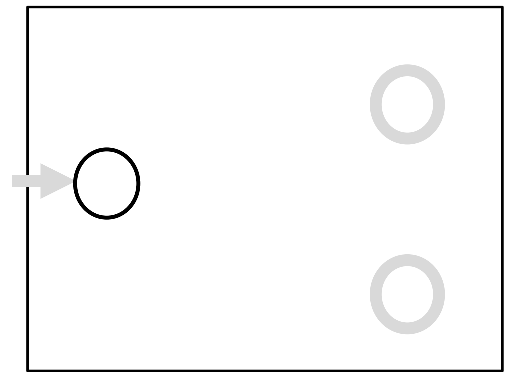
N_A



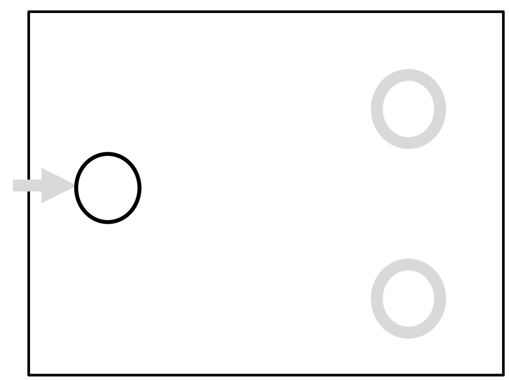
N_B

Inductive Step

Case A ∪ B:



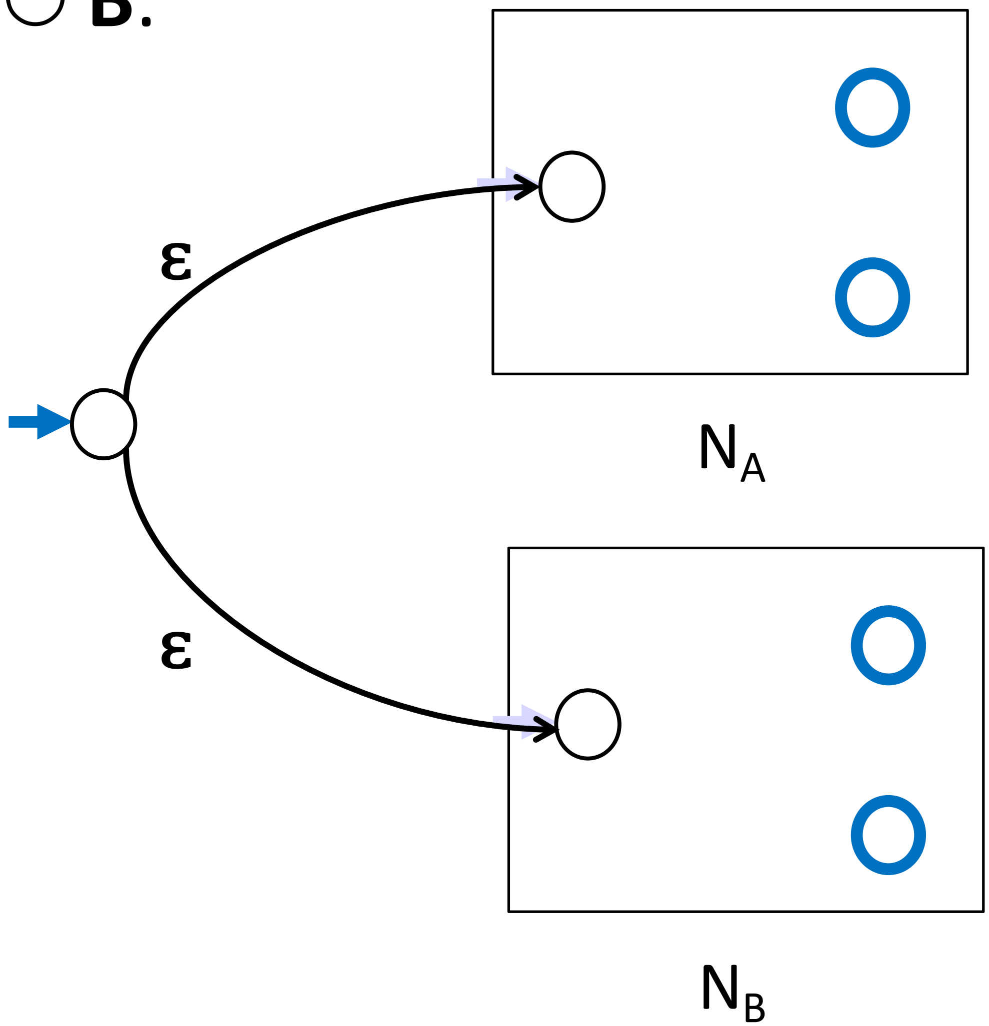
N_A



N_B

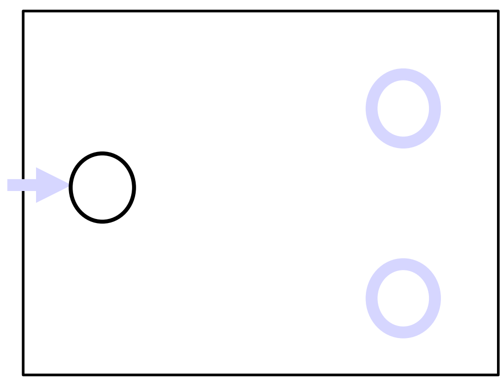
Inductive Step

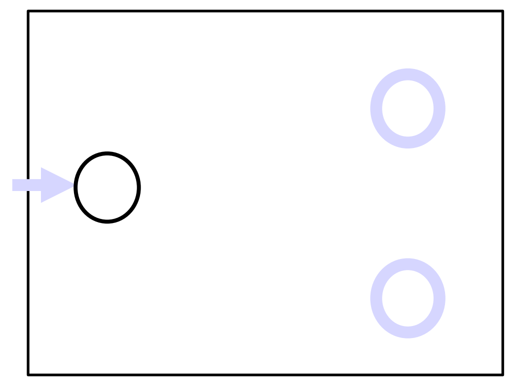
Case A ∪ B:



Inductive Step

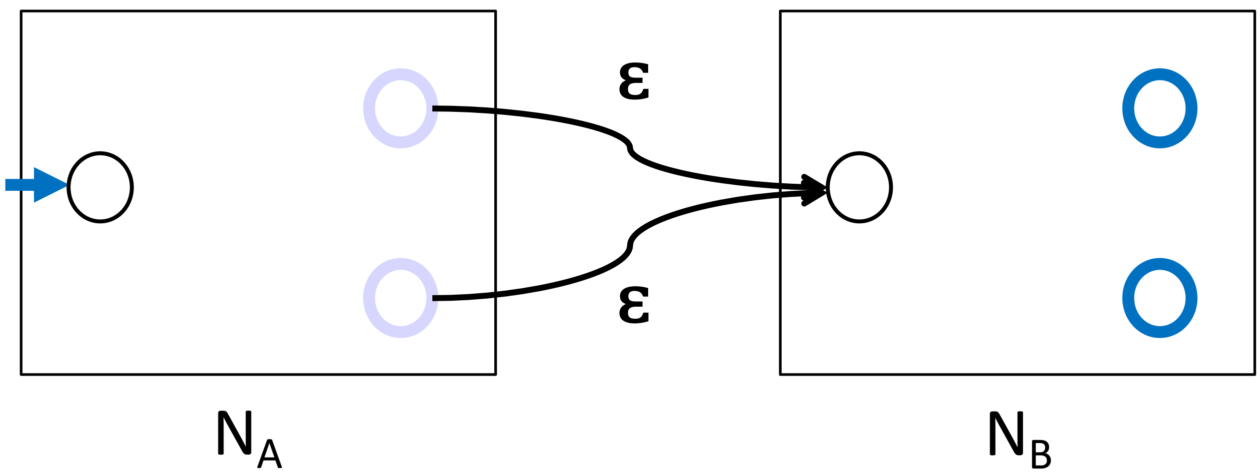
Case AB:





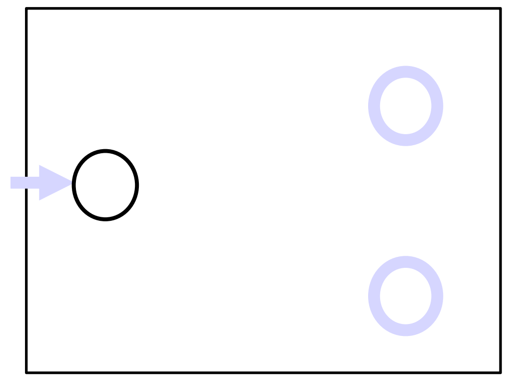
Inductive Step

Case AB:



Inductive Step

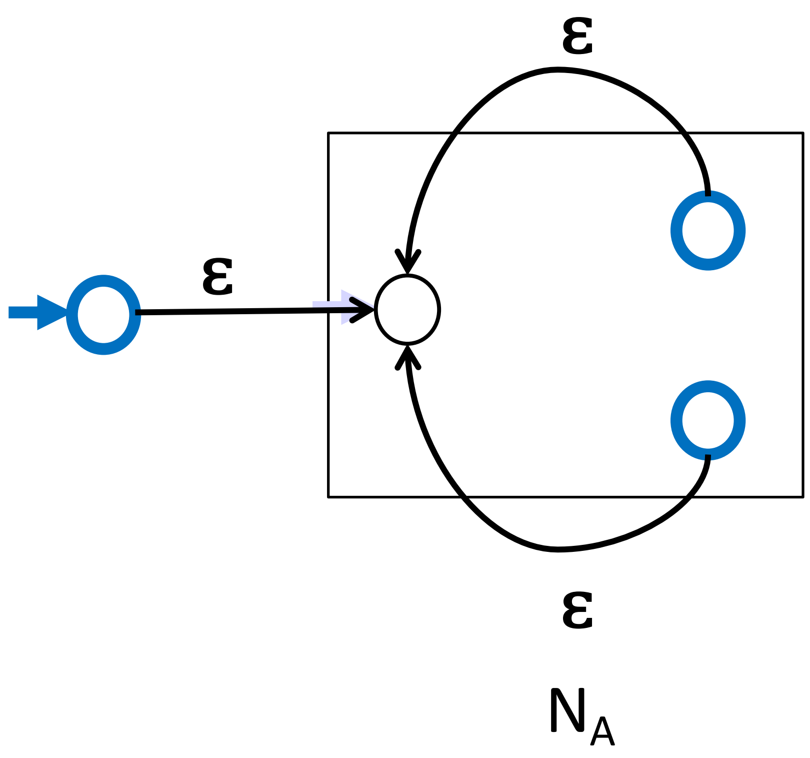
Case A*



N_A

Inductive Step

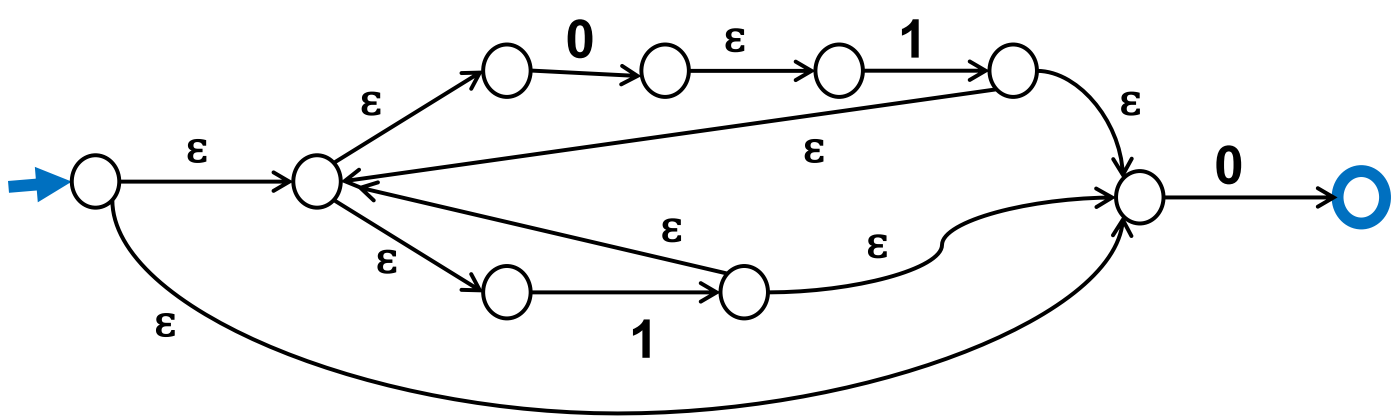
Case A*



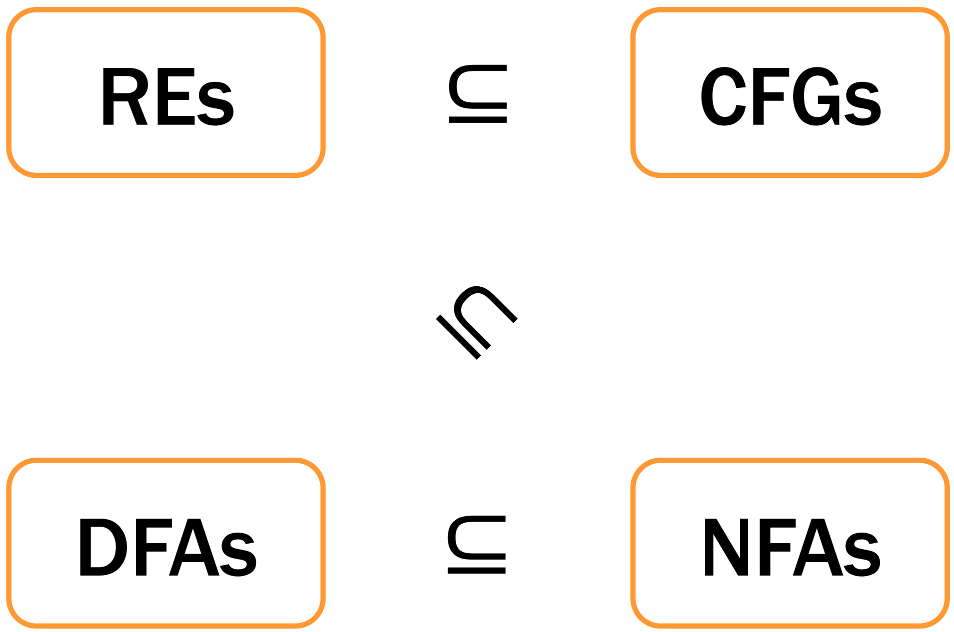
Build an NFA for $(01 \cup 1)^*0$

Solution

(01 ∪1)*0



The story so far...



NFAs and DFAs

Every DFA is an NFA

- DFAs have requirements that NFAs don't have

Can NFAs recognize more languages?

NFAs and DFAs

Every DFA **is** an NFA

– DFAs have requirements that NFAs don't have

Can NFAs recognize more languages? No!

Theorem: For every NFA there is a DFA that recognizes exactly the same language

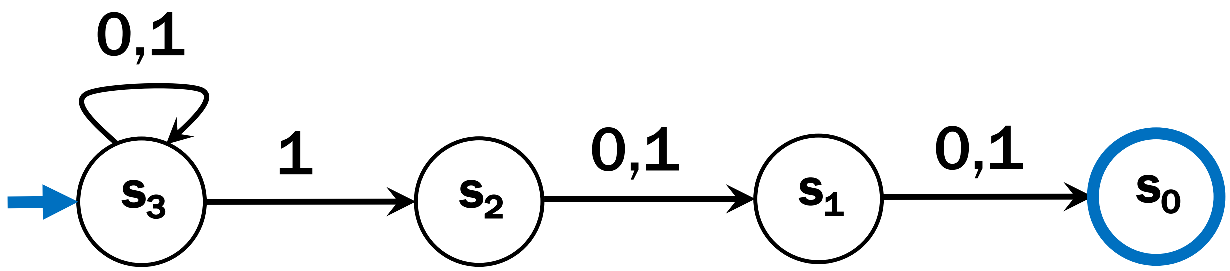
Three ways of thinking about NFAs

- Outside observer: Is there a path labeled by x from the start state to some final state?

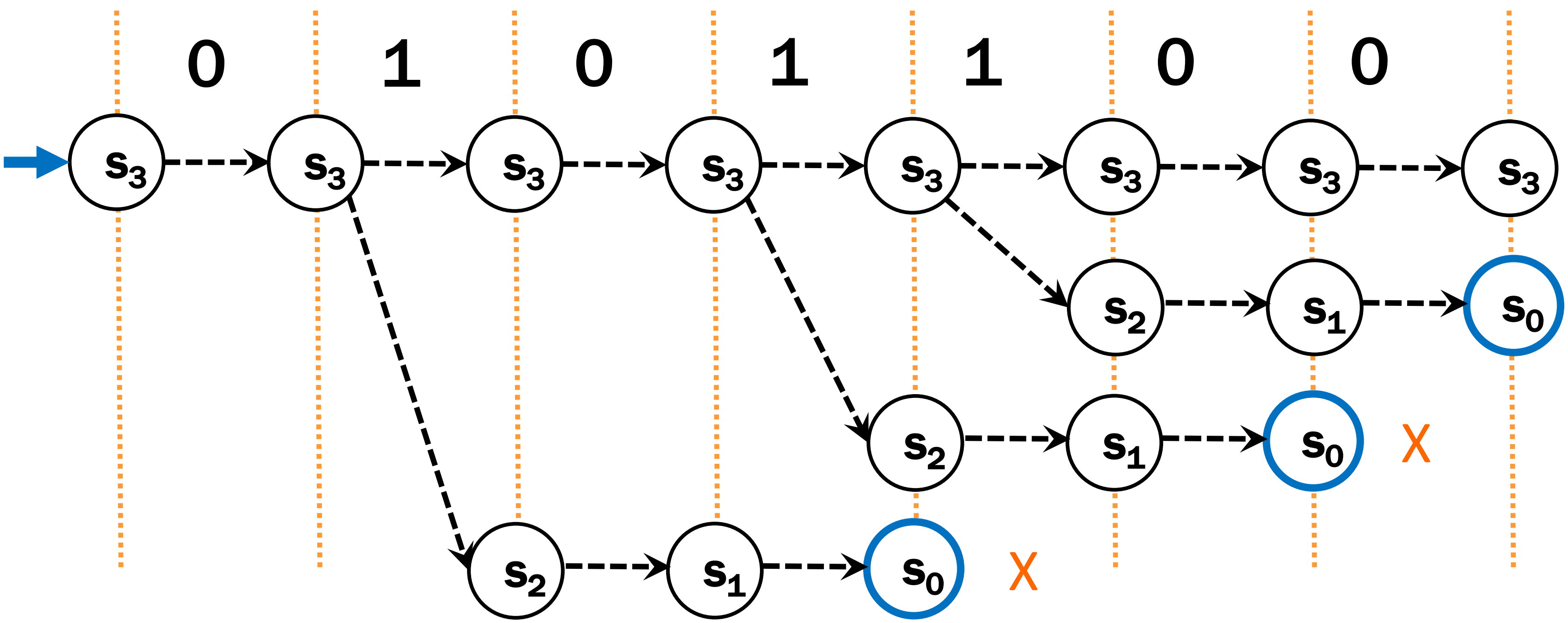
- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)

- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

Parallel Exploration view of an NFA



Input string 0101100



Conversion of NFAs to a DFAs

- **Construction Idea:**
  - **The DFA keeps track of ALL states reachable in the NFA along a path labeled by the input so far**

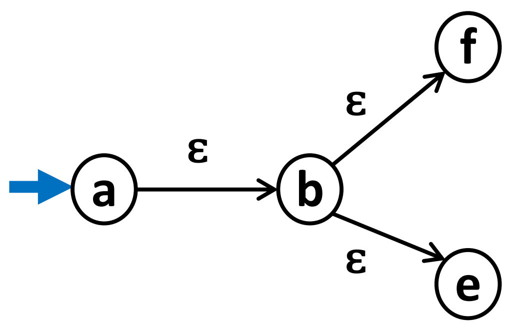
    (Note: not all *paths*; all *last states* on those paths.)

  - **There will be one state in the DFA for each *subset* of states of the NFA that can be reached by some string**

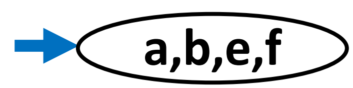
Conversion of NFAs to a DFAs

New start state for DFA

- The set of all states reachable from the start state of the NFA using only edges labeled ε



NFA

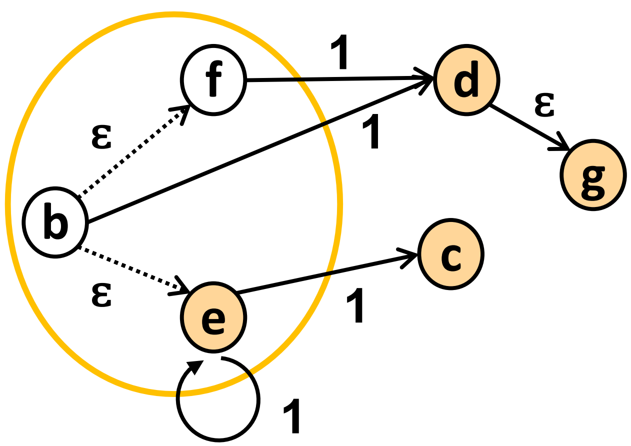


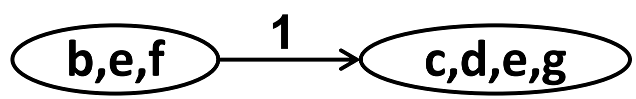
DFA

Conversion of NFAs to a DFAs

For each state of the DFA corresponding to a set S of states of the NFA and each symbol s

- Add an edge labeled s to state corresponding to T, the set of states of the NFA reached by
  - starting from some state in S, then
  - following one edge labeled by s, and then following some number of edges labeled by **ε**
- T will be ∅ if no edges from S labeled s exist

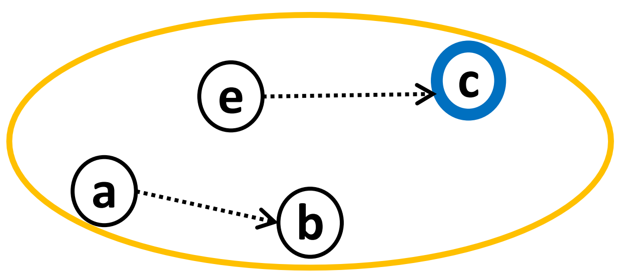




Conversion of NFAs to a DFAs

Final states for the DFA

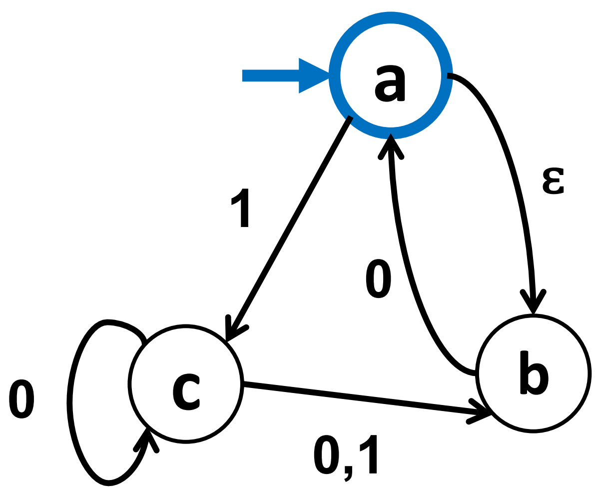
- All states whose set contain some final state of the NFA



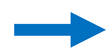
NFA

DFA

Example: NFA to DFA

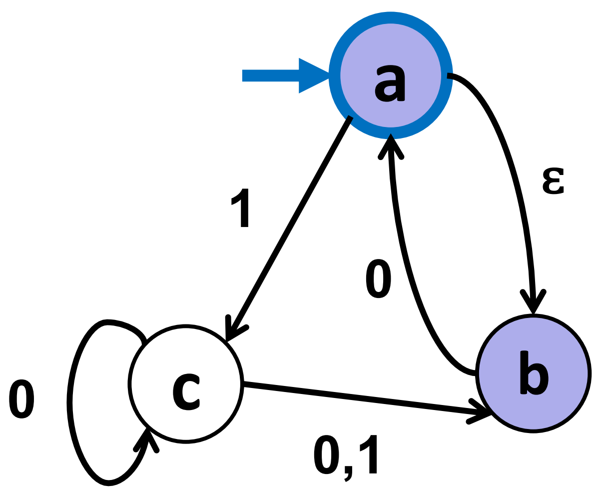


NFA

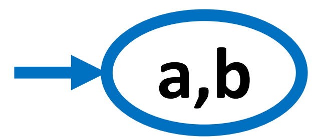


DFA

Example: NFA to DFA



NFA



DFA

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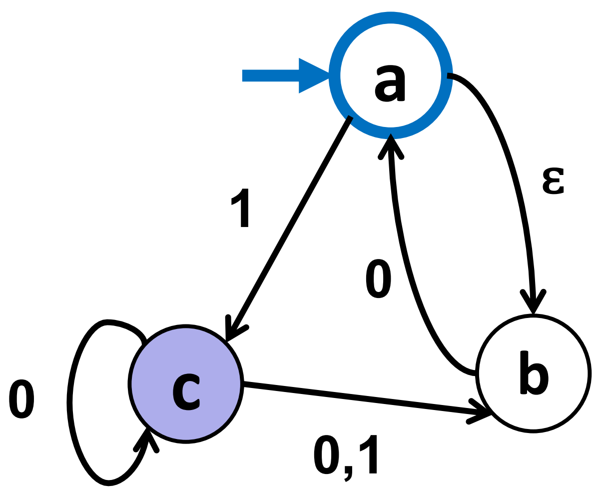
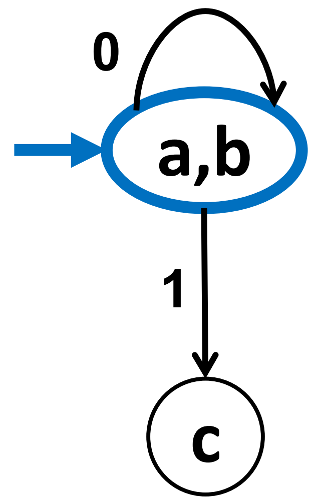


NFA



DFA

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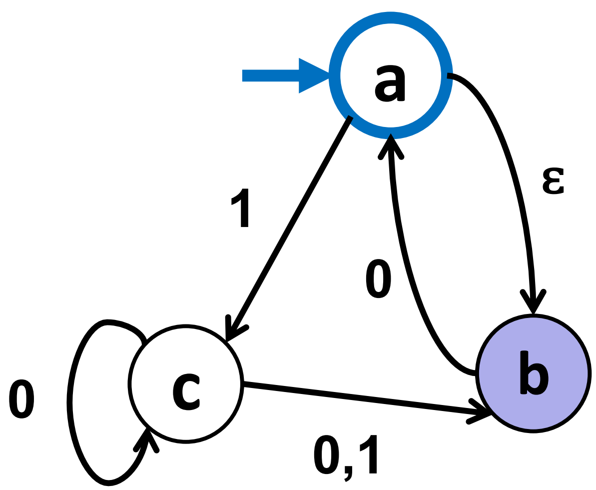
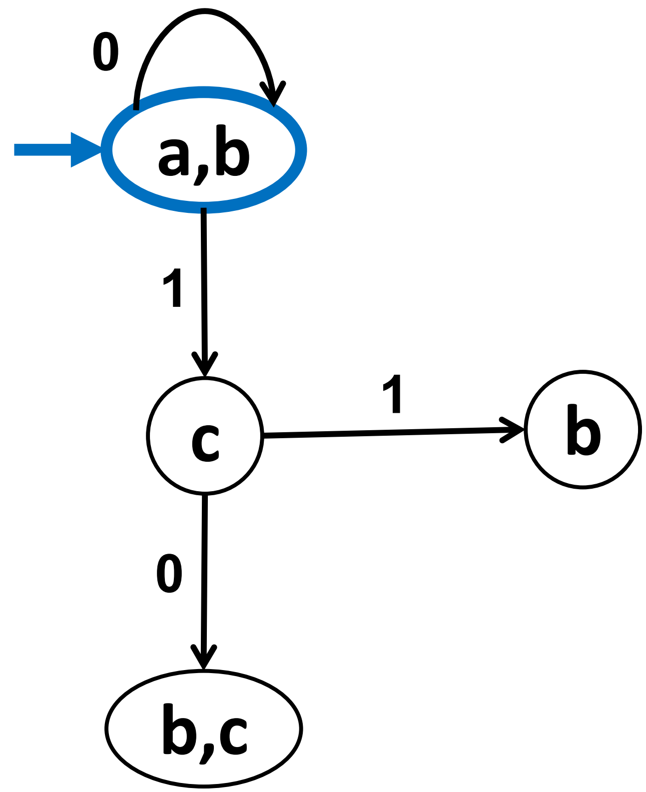
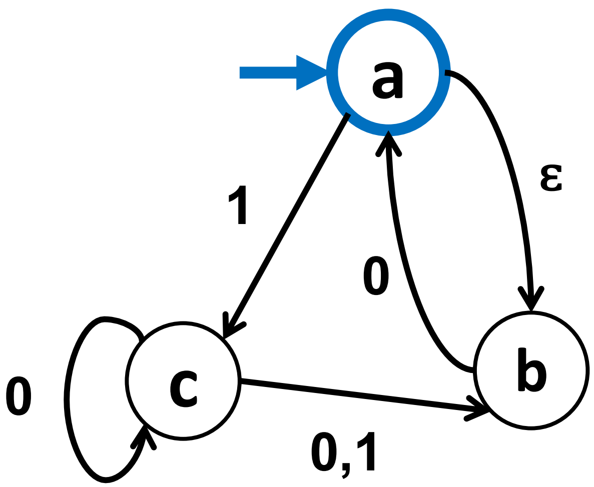


NFA

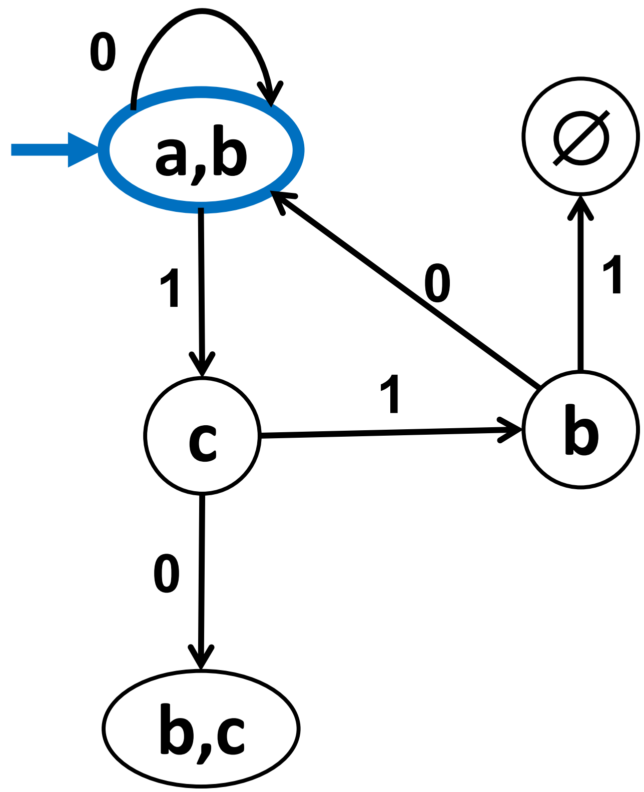


DFA

Example: NFA to DFA

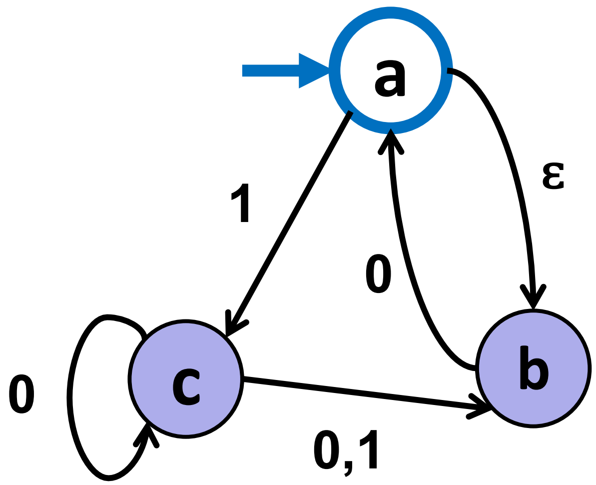


NFA

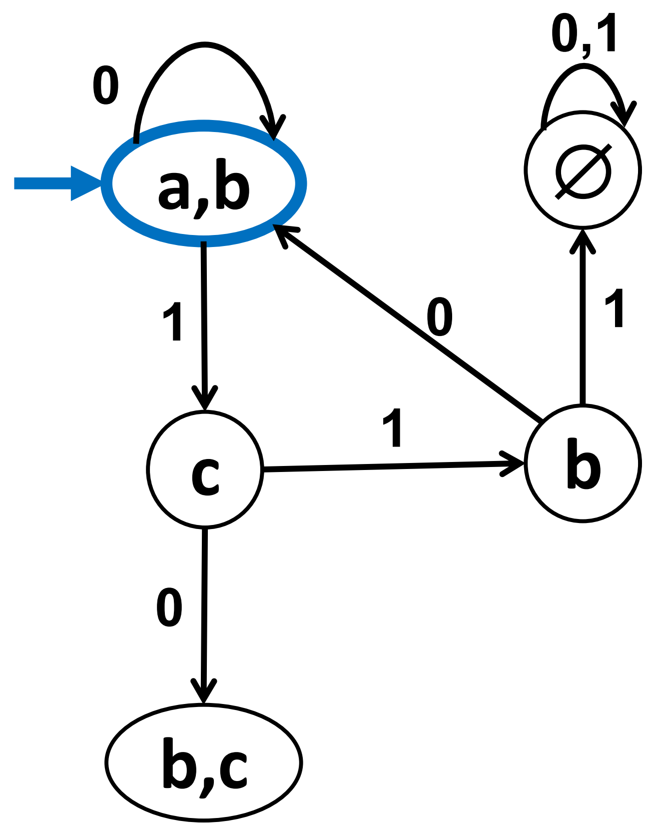


DFA

Example: NFA to DFA

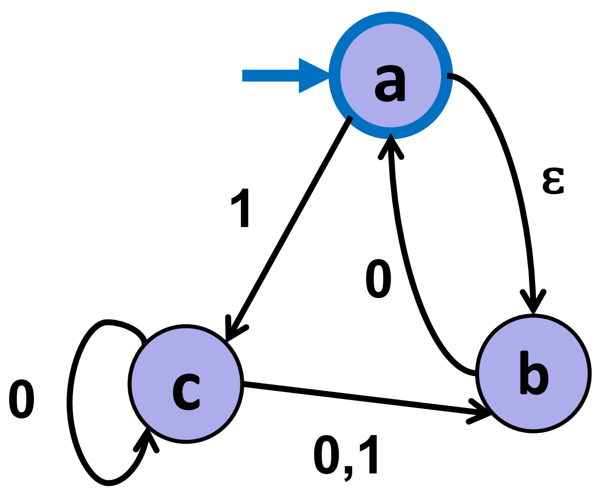


NFA

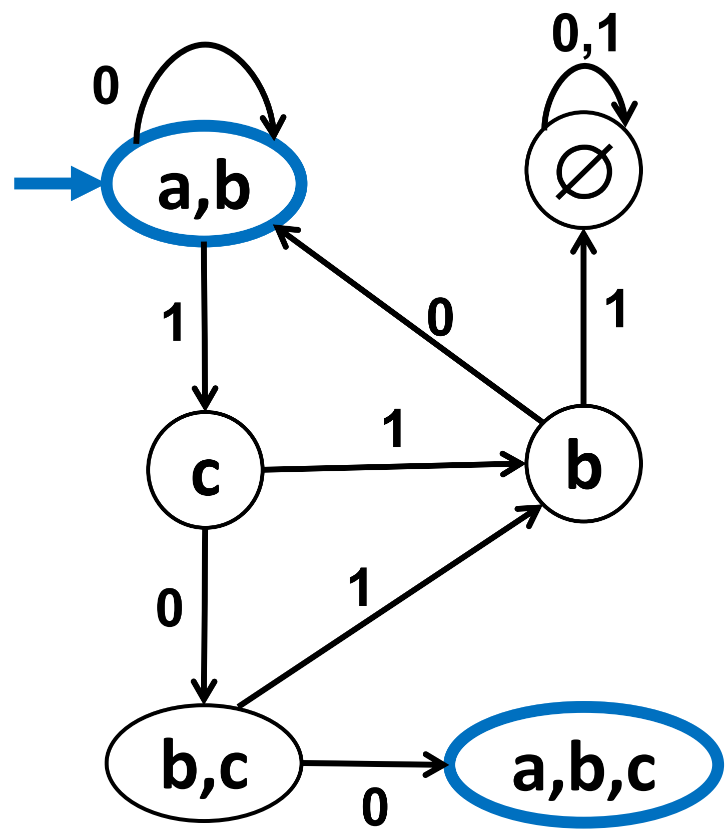


DFA

Example: NFA to DFA

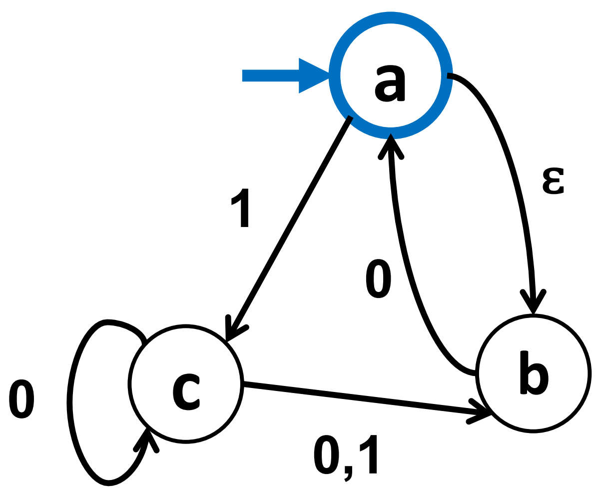


NFA

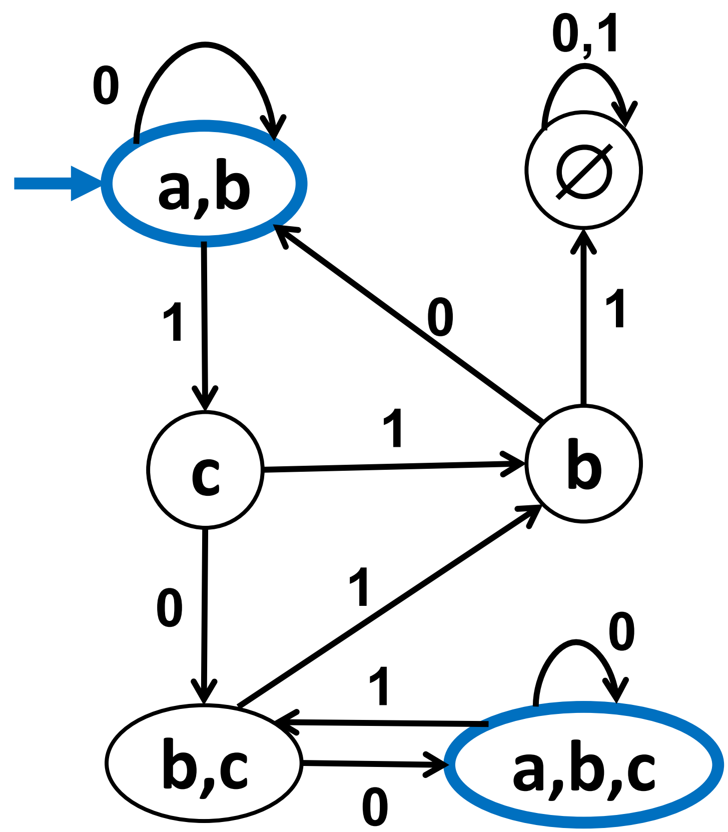


DFA

Example: NFA to DFA

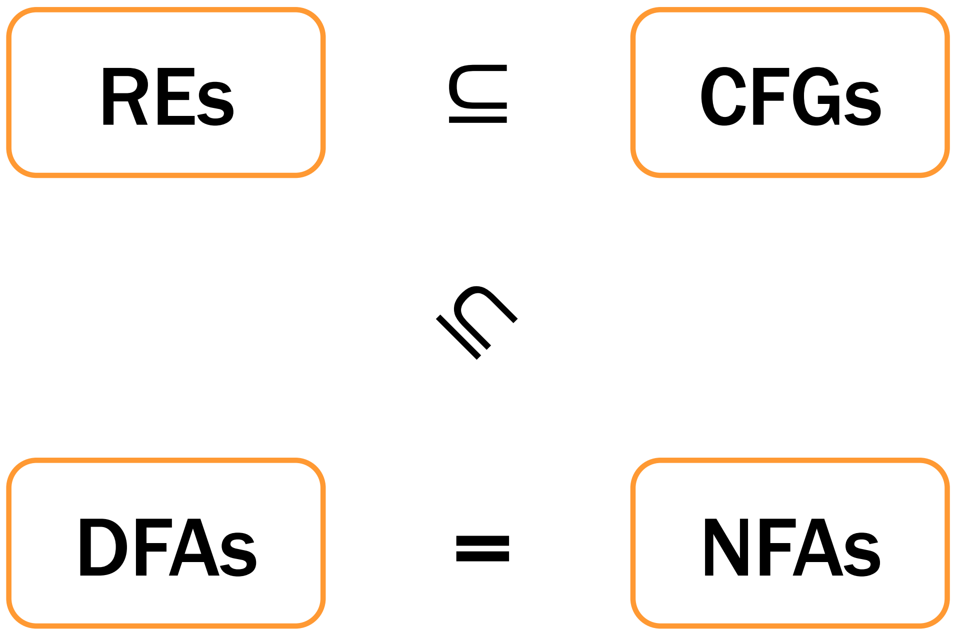


NFA



DFA

The story so far...



Regular expressions $\subseteq$ NFAs $\equiv$ DFAs

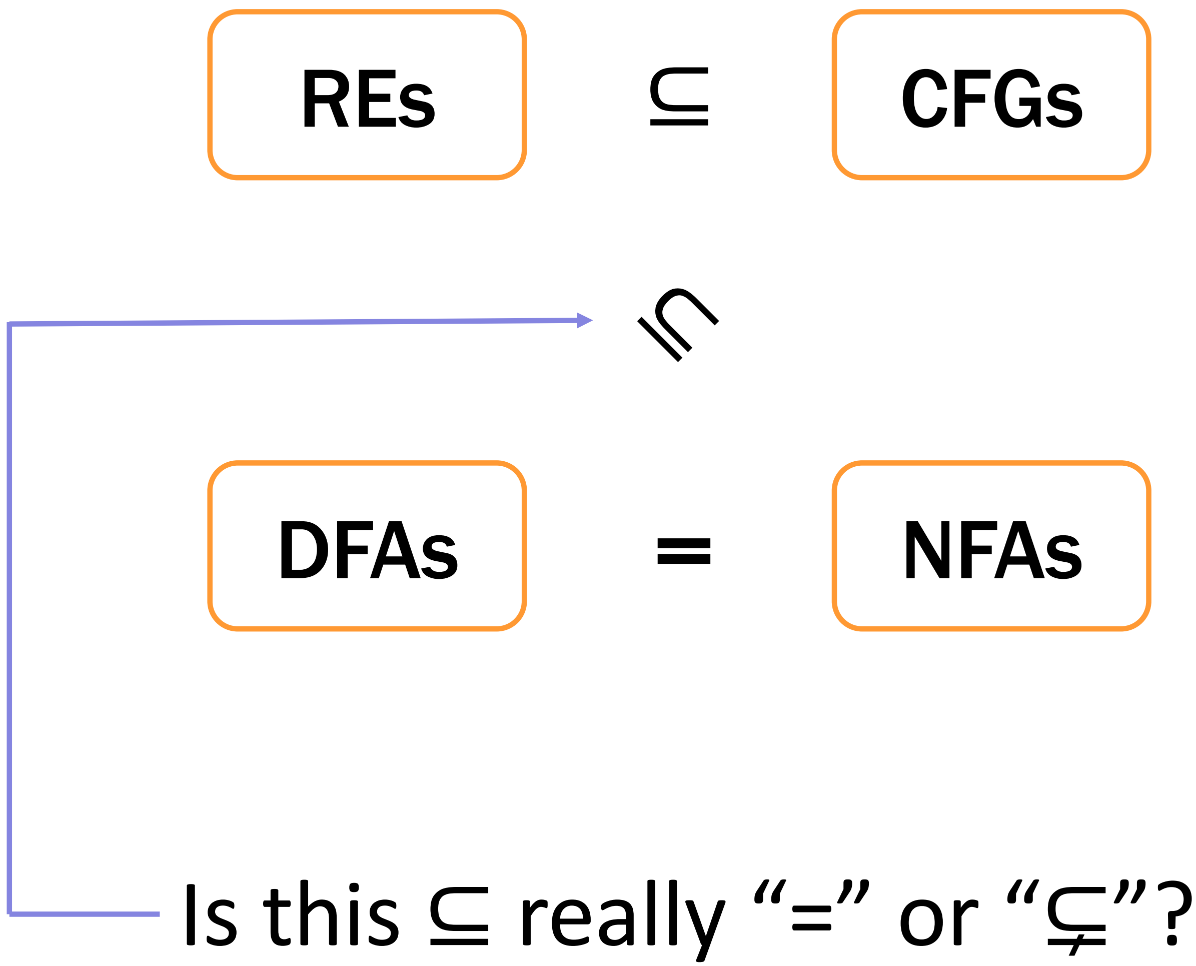
We have shown how to build an optimal DFA for every regular expression

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Thus, we could now implement a RegExp library

- most RegExp libraries actually simulate the NFA
- (even better: one can combine the two approaches: apply DFA minimization lazily while simulating the NFA)

The story so far...



Regular expressions ≡ NFAs ≡ DFAs

Theorem: For any NFA, there is a regular expression that accepts the same language

Corollary: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

You need to know these facts

- the construction for the Theorem is sketched below but you will not be tested on it

New Machinery: Generalized NFAs

- Like NFAs but allow
  - parallel edges (between the same pair of states)
  - regular expressions as edge labels

    NFAs already have edges labeled **ε** or ***a***

- Machine can follow an edge labeled by **A** by reading a <u>string of input characters</u> in the language of A
  - (if A is ***a*** or ε, this matches the original definition, but we now allow REs built with recursive steps.)

New Machinery: Generalized NFAs

- Like NFAs but allow
  - parallel edges
  - regular expressions as edge labels

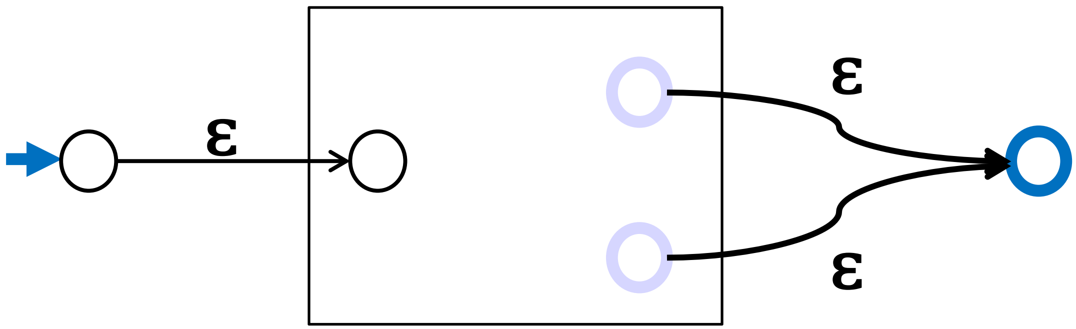
    NFAs already have edges labeled **ε** or ***a***

- The label of a path is now the concatenation of the regular expressions on those edges, making it a regular expression

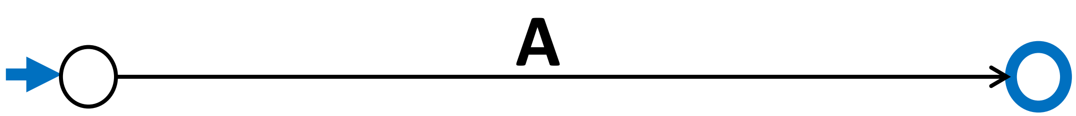
- Def: A string x is accepted by a generalized NFA iff there is a *path* from start to final state labeled by a regular expression whose language **contains** x

Construction Idea

Add new start state and final state

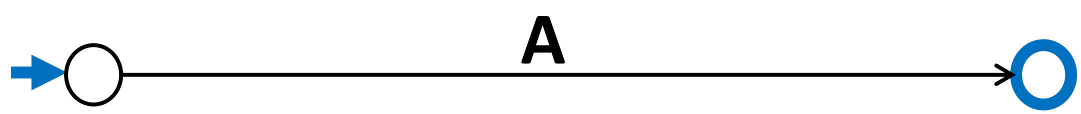


Then delete the original states one by one, adding edges to keep the same language, until the graph looks like:



Starting from an NFA

Then delete the original states one by one, adding edges to **keep the same language**, until the graph looks like:

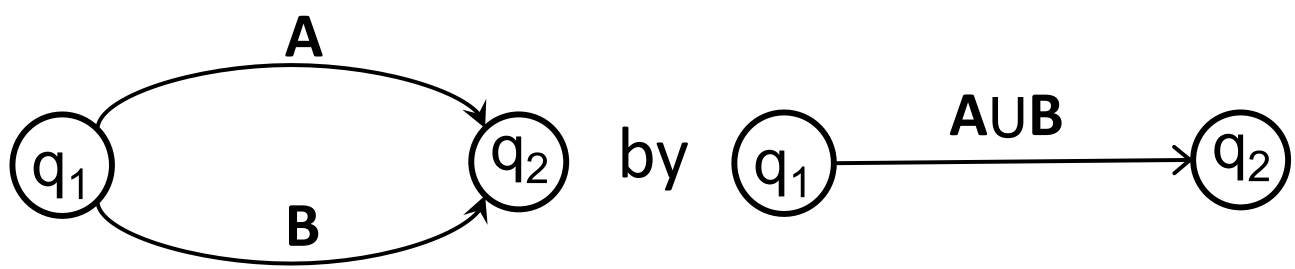


Final graph has only one path to the accepting state, which is labeled by A, so it accepts iff x is in the language of A

Thus, A is a regular expression with the same language as the original NFA.

Only two simplification rules

- **Rule 1**: For any two states q_1 and q_2 with parallel edges (possibly $q_1=q_2$), replace

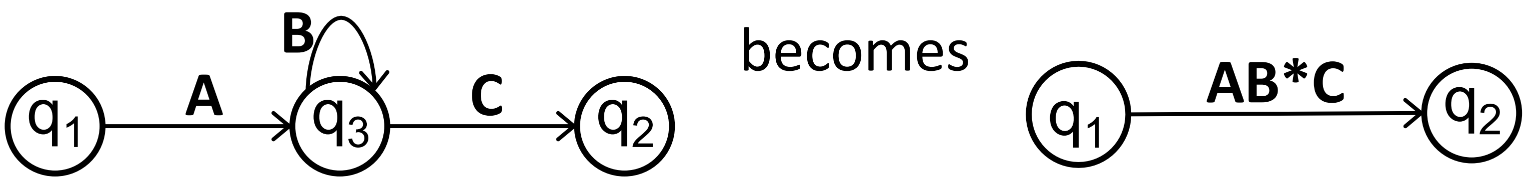


If the machine would have used the edge labeled A by consuming an input x in the language of A, it can instead use the edge labeled A∪B.

Furthermore, this new edge does not allow transitions for any strings other than those that matched A or B.

Only two simplification rules

- **Rule 2**: Eliminate non-start/accepting state q_3 by creating direct edges that skip q_3

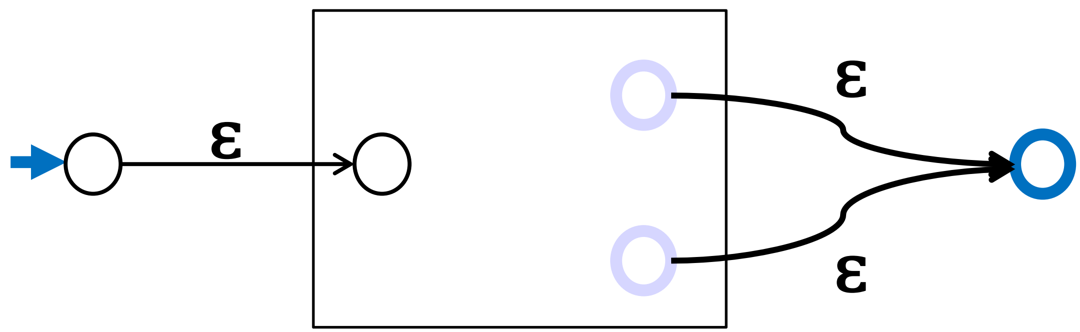


for *every* pair of states q_1, q_2 (even if $q_1=q_2$)

Any path from q_1 to q_2 would have to match AB^nC for some n (the number of times the self loop was used), so the machine can use the new edge instead. New edge *only* allows strings that were allowed before.

Construction Overview

Add new start state and final state



While the box contains some state s:

for all states r, t with (r, s) and (s, t) in E:

create a direct edge (r, t) by Rule 2

delete s (no longer needed)

merge all parallel edges by Rule 1

Construction Overview

While the box contains some state s:

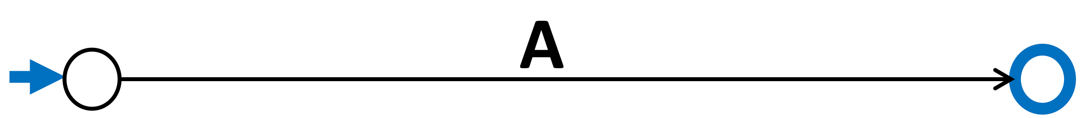
for all states r, t with (r, s) and (s, t) in E:

create a direct edge (r, t) by Rule 2

delete s (no longer needed)

merge all parallel edges by Rule 1

When the loop exits, the graph looks like this:

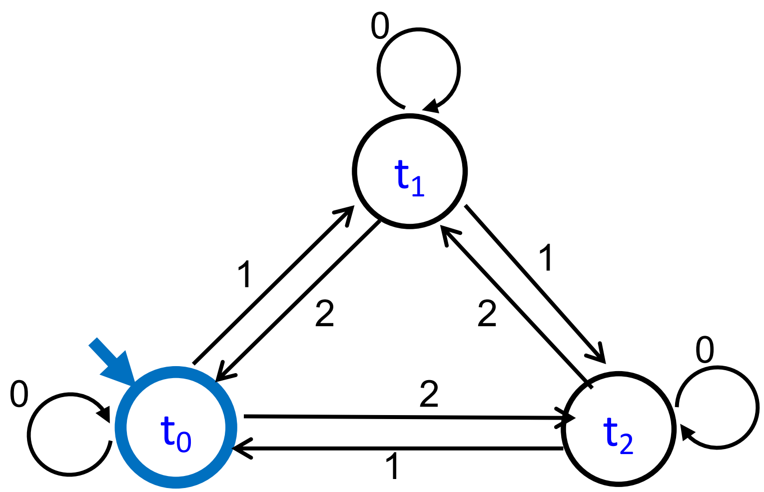


A is a regular expression with the same language as the original NFA.

Converting an NFA to a regular expression

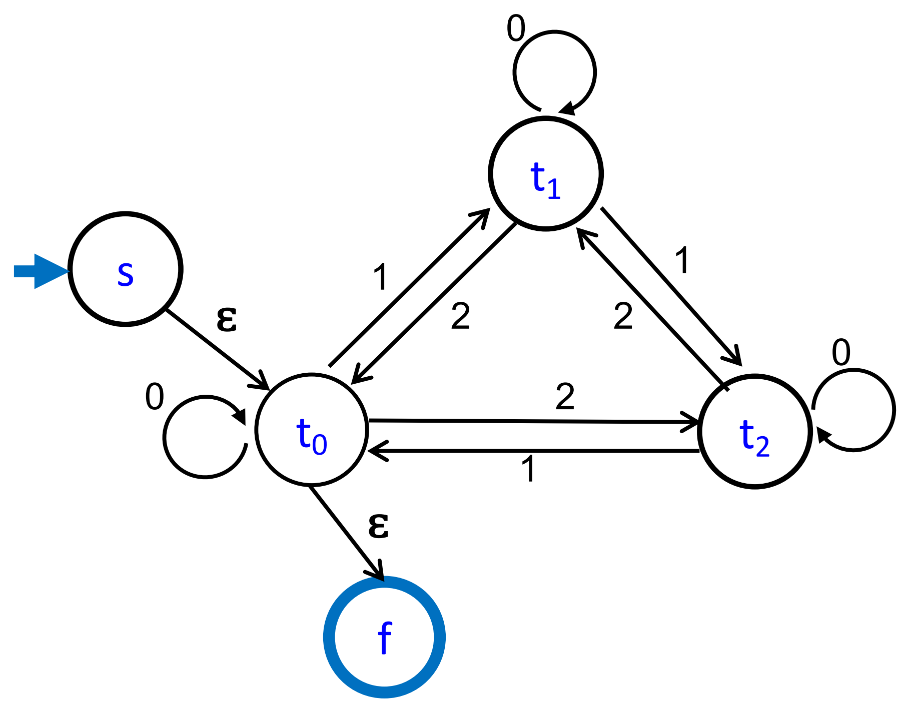
Consider the DFA for the mod 3 sum

- Accept strings from {0,1,2}* where the digits mod 3 sum of the digits is 0



Splicing out a state t_1

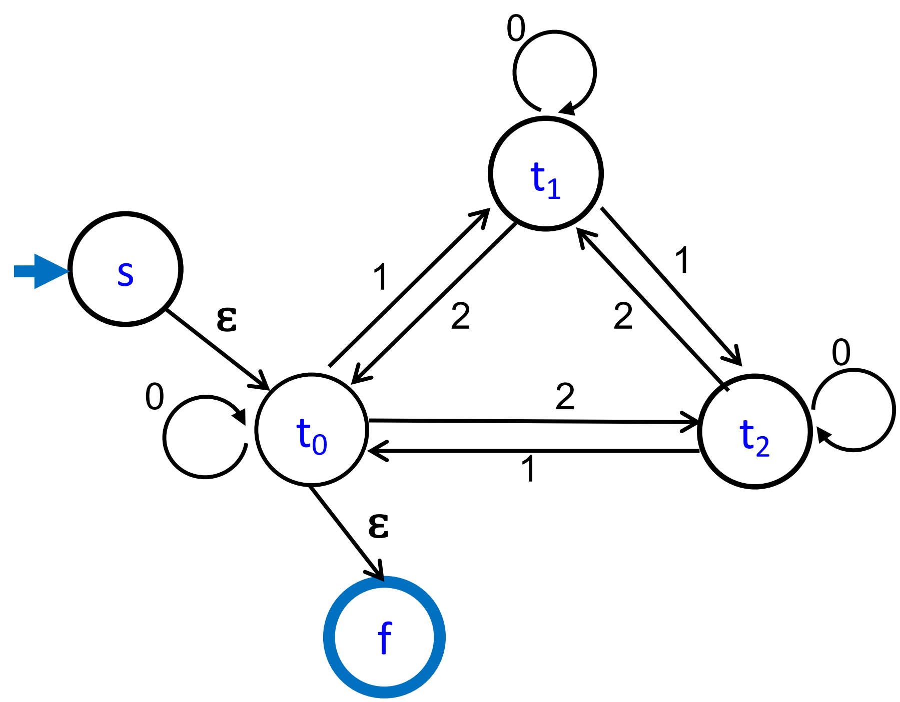
Create direct edges between neighbors of t_1 (so that we can delete it afterward)



Splicing out a state t_1

Regular expressions to add to edges

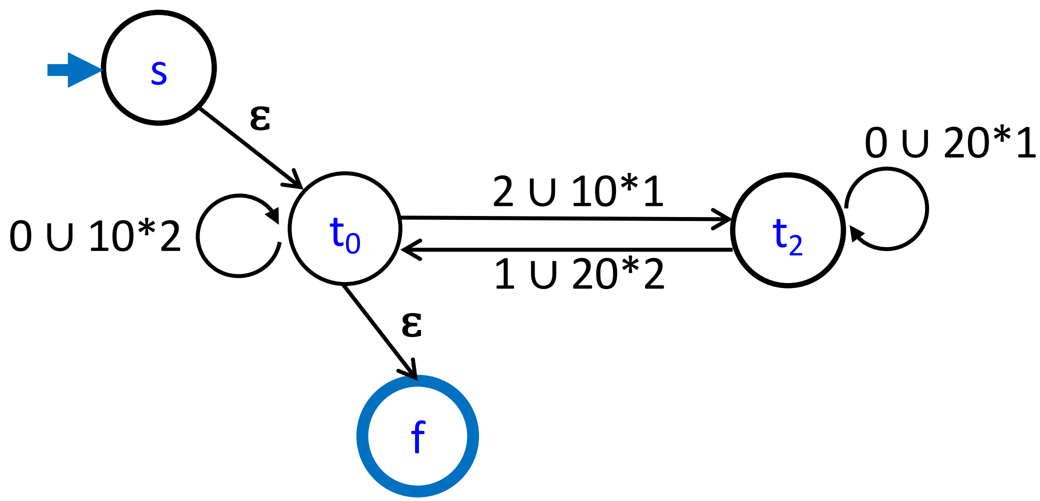
$t_0 \rightarrow t_1 \rightarrow t_0$: 10*2
$t_0 \rightarrow t_1 \rightarrow t_2$: 10*1
$t_2 \rightarrow t_1 \rightarrow t_0$: 20*2
$t_2 \rightarrow t_1 \rightarrow t_2$: 20*1



Splicing out a state t_1

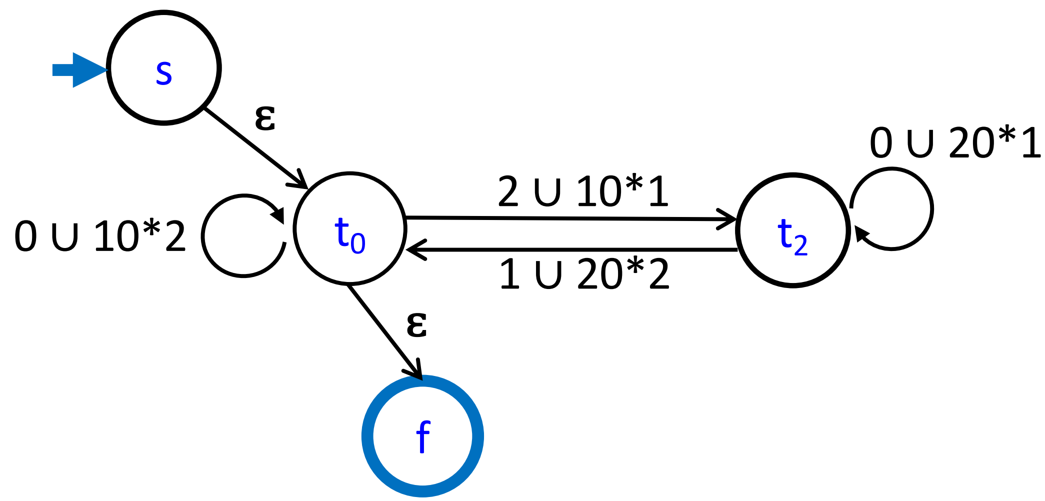
Delete t_1 now that it is redundant

$t_0 \rightarrow t_1 \rightarrow t_0$:   10*2
$t_0 \rightarrow t_1 \rightarrow t_2$:   10*1
$t_2 \rightarrow t_1 \rightarrow t_0$:   20*2
$t_2 \rightarrow t_1 \rightarrow t_2$:   20*1



Splicing out a state t_1

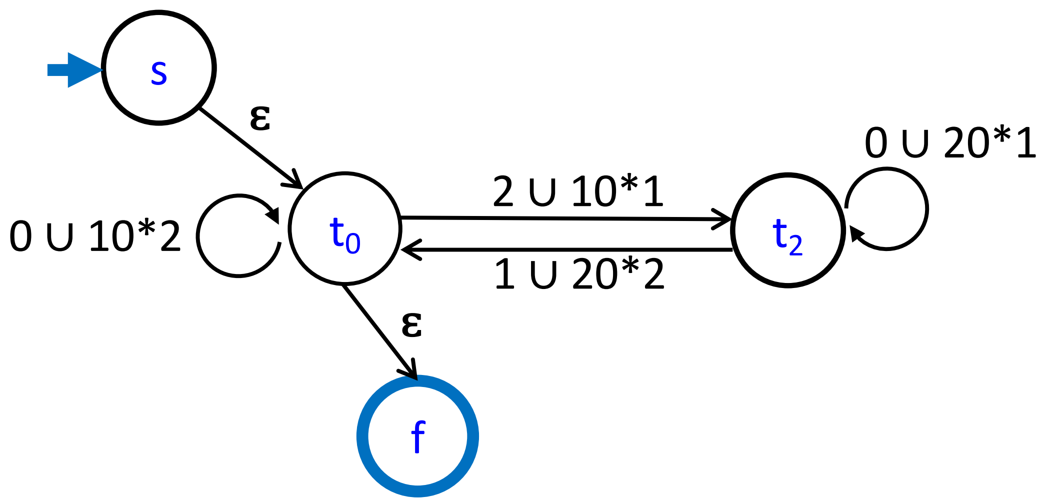
Create direct edges between neighbors of t_2
(so that we can delete it afterward)



Splicing out a state t_1

Regular expressions to add to edges

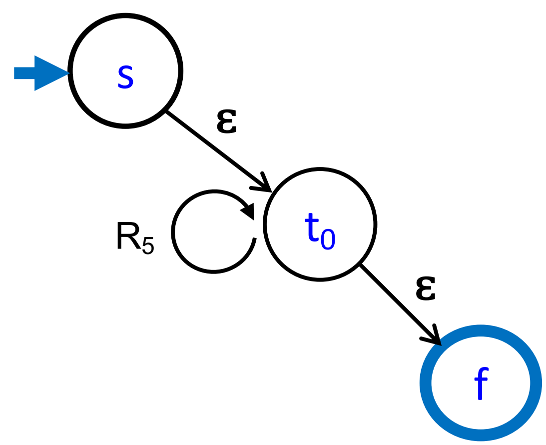
R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$



Splicing out state t_2 (and then t_0)

Delete t_2 now that it is redundant

R_1: 0 ∪ 10*2
R_2: 2 ∪ 10*1
R_3: 1 ∪ 20*2
R_4: 0 ∪ 20*1

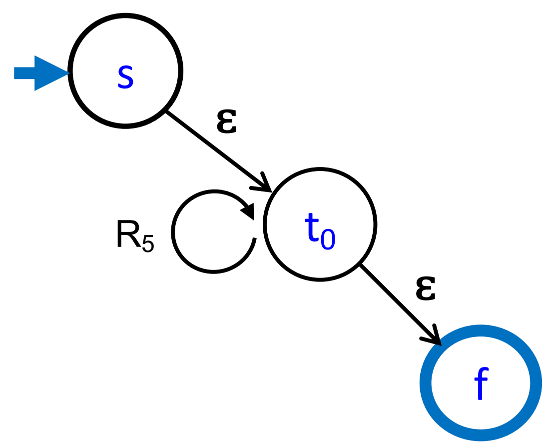


R_5: $R_1 \cup R_2R_4{}^*R_3$

Splicing out state t_2 (and then t_0)

Create direct (s,f) edge so we can delete t_0

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$

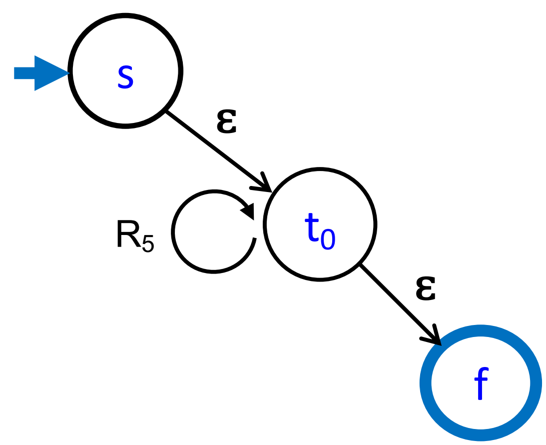


Splicing out state t_2 (and then t_0)

Regular expressions to add to edges

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$

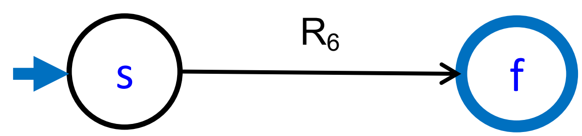
$t_0 \rightarrow t_1 \rightarrow t_0$: $R_5{}^*$



Splicing out state t_2 (and then t_0)

Delete t_0 now that it is redundant

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4^*R_3$

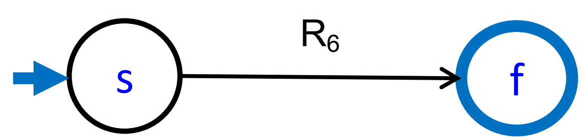


R_6: R_5^*

Splicing out state t_2 (and then t_0)

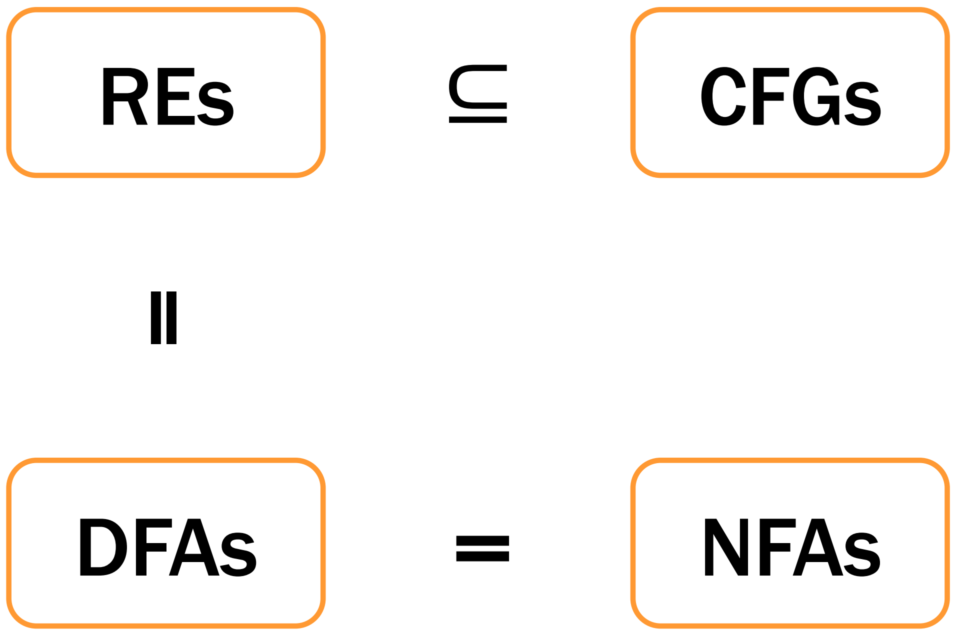
Regular expressions to add to edges

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$
R_6: $R_5{}^*$

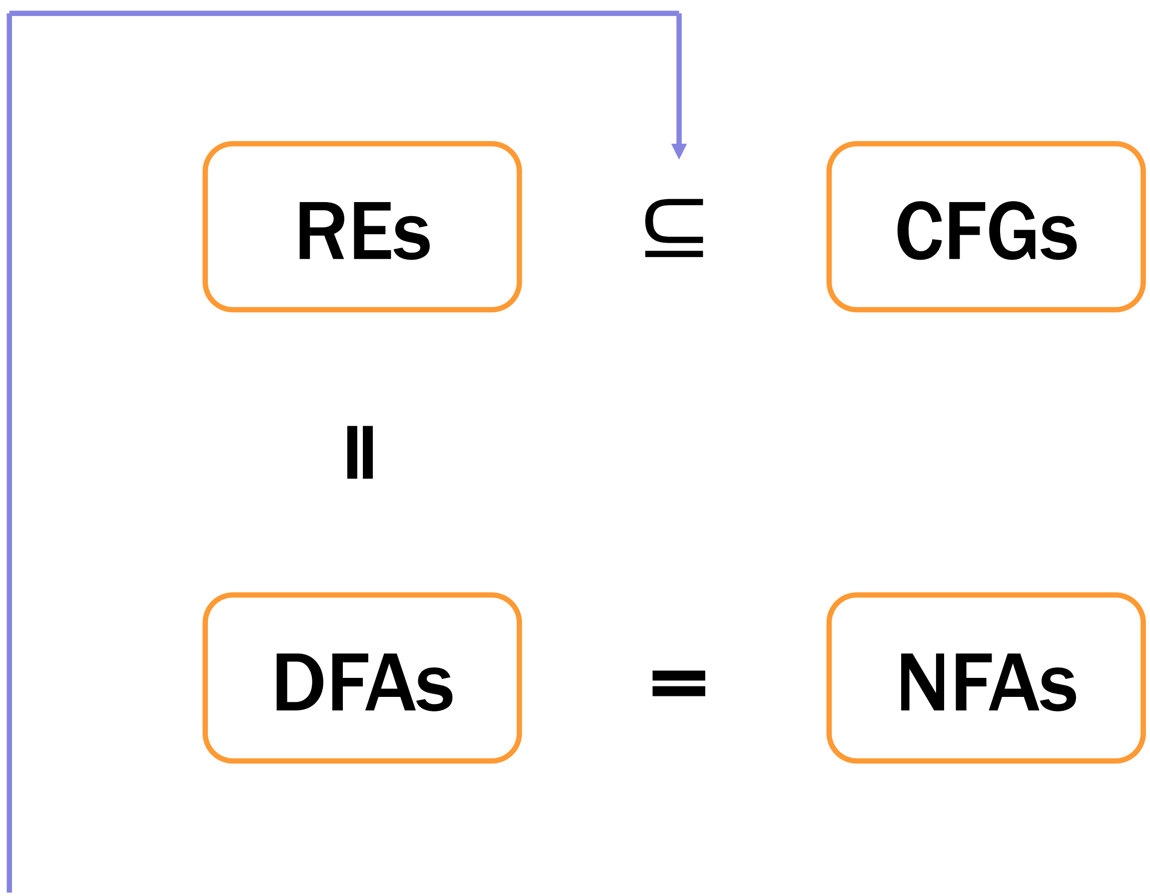


Final regular expression: R_6 =
$(0 \cup 10^*2 \cup (2 \cup 10^*1)(0 \cup 20^*1)^*(1 \cup 20^*2))^*$

The story so far...



The story so far…



<u>Next time</u>: Is this ⊆ really "=" or "⊊"?