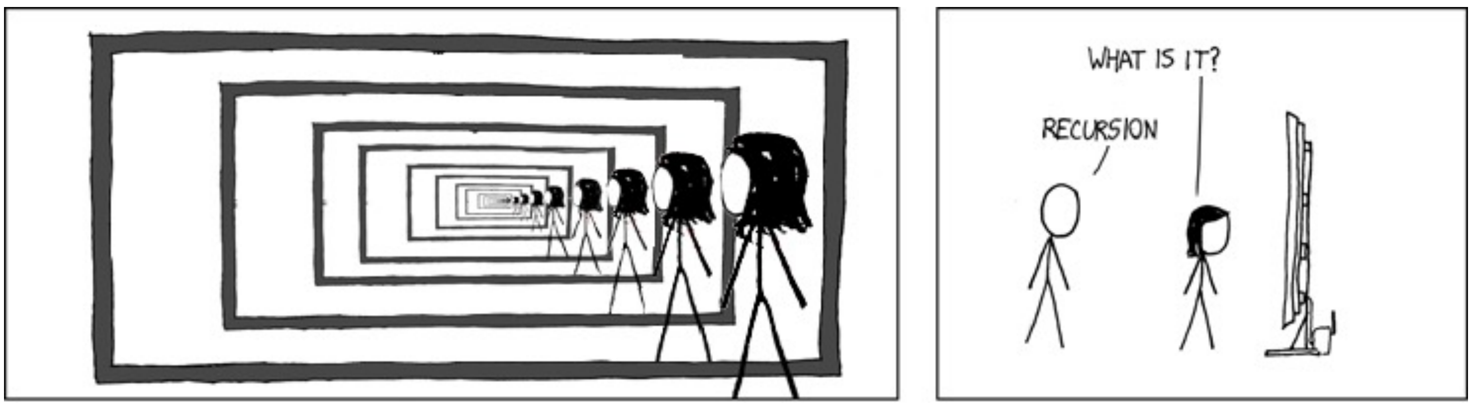
CSE 311: Foundations of Computing

Lecture 18: Recursively Defined Sets & Structural Induction



Midterm

- Friday in class
- Covers material up to end of ordinary induction
- Closed book, closed notes
  - will provide reference sheets
- No calculators
  - arithmetic is intended to be straightforward
  - (only a small point deduction anyway)

Midterm

- 5 problems covering:
  - Logic / English translation
  - Circuits / Boolean algebra / normal forms
  - Solving modular equations
  - Induction
  - Set theory
  - (all English proofs)

Review Session
Thu at 1:30
in ECE 125

- 10 minutes per problem
  - write quickly
  - focus on the overall structure of the solution

Last time: Recursive definitions of functions

- $0! = 1$; $(n+1)! = (n+1) \cdot n!$ **for all** $n \geq 0$**.**

- $F(0) = 0$; $F(n+1) = F(n) + 1$ **for all** $n \geq 0$**.**

- $G(0) = 1$; $G(n+1) = 2 \cdot G(n)$ **for all** $n \geq 0$**.**

- $H(0) = 1$; $H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$**.**

Last time: Recursive definitions of functions

- Recursive functions allow general computation
  - saw examples not expressible with simple expressions
- So far, we have considered only simple data
  - inputs and outputs were just integers
- We need general data as well…
  - these will also be described *recursively*
  - will allow us to describe data of real programs

Recursive Definitions of Sets (Data)

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step: If** $x \in S$**, then** $x + 2 \in S$
- **Exclusion Rule: Every element in** S **follows from the basis step and a finite number of recursive steps.**

We need the exclusion rule because otherwise $S=\mathbb{N}$ would satisfy the other two parts. However, we won't always write it down on these slides.

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Basis: $(0, 0) \in S$, $(1, 1) \in S$

Recursive: If $(n-1, x) \in S$ and $(n, y) \in S$, then $(n+1, x + y) \in S$.

?

Recursive Definitions of Sets

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Even numbers

Basis: $0 \in S$

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Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Fibonacci numbers

Basis: $(0, 0) \in S$, $(1, 1) \in S$

Recursive: If $(n-1, x) \in S$ and $(n, y) \in S$, then $(n+1, x + y) \in S$.

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ* of *strings* over the alphabet Σ
  - example: {0,1}* is the set of *binary strings*

    0, 1, 00, 01, 10, 11, 000, 001, … and “”

- Σ* is defined recursively by
  - Basis: $\varepsilon \in \Sigma^*$ (ε is the empty string, i.e., “”)
  - Recursive: if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same when read backwards and forwards

Basis:

ε is a palindrome

any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome,

then apa is a palindrome for every $a \in \Sigma$

Functions on Recursively Defined Sets (on Σ^*)

Length:

defined by cases

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = \text{len}(w) + 1$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

concat(x,y) or $x \bullet y$
defined by cases
on the shape of **y**

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = a \bullet w^R$ for $w \in \Sigma^*$, $a \in \Sigma$

reverse(x) or x^R

Number of c's in a string:

$\#_c(\varepsilon) = 0$

$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

more cases (3 total)
separate c vs $a \neq c$

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, **then** $0 \bullet x \in S$

If $x \in S$, **then** $x1 \in S$

Those have no 1s before 0s.
But is that every such string?

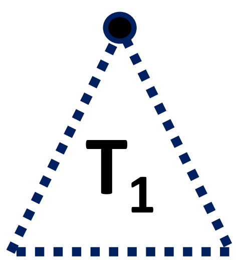
Rooted Binary Trees

- **Basis:** • is a rooted binary tree

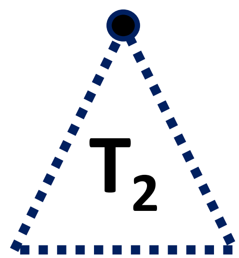
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If

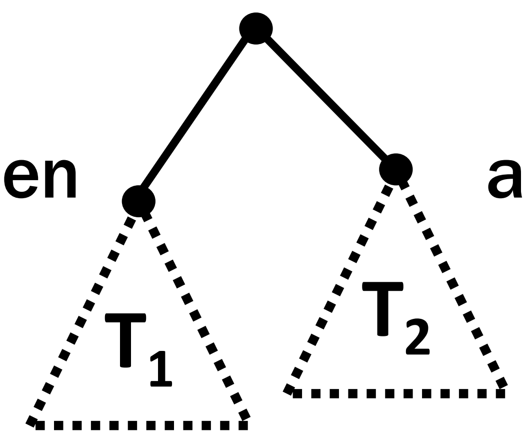


and



are rooted binary trees,

then



also is a rooted binary tree.

Rooted Binary Trees in Java

```
public static class BinaryTree {
  static BinaryTree LEAF = ...;
  public BinaryTree(
     BinaryTree T1, BinaryTree T2) {
   ...
  }
}
```

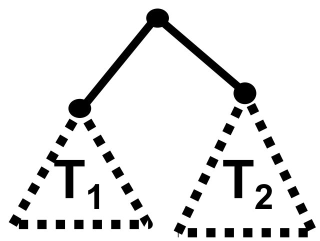
Create a binary tree with

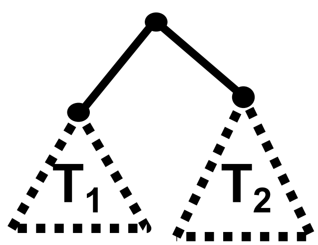
`BinaryTree.LEAF` **or**

`new BinaryTree(T1, T2)`

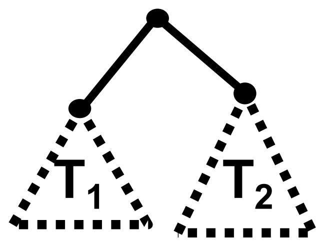
Recursively-defined Sets
translate natural into Java classes

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size () = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)

- height(•) = 0
- height ()=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Functions on Rooted Binary Trees in Java

- size(•) = 1
- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)

```
public int size(BinaryTree T) {
  if (T == BinaryTree.LEAF) {
    return 1;
  } else {
    return 1 + size(T.left()) + size(T.right());
  }
}
```

Recursive Functions translate natural into Java functions

Recursive Sets translate natural into Java classes

Last time: Recursive definitions of functions

- Before, we considered only simple data
  - inputs and outputs were just integers
- Proved facts about those functions with **induction**
  - $n! \leq n^n$
  - $f_n < 2^n$ and $f_n \geq 2^{n/2-1}$
- How do we prove facts about functions that work with more complex (recursively defined) data?
  - we need a more sophisticated form of induction

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

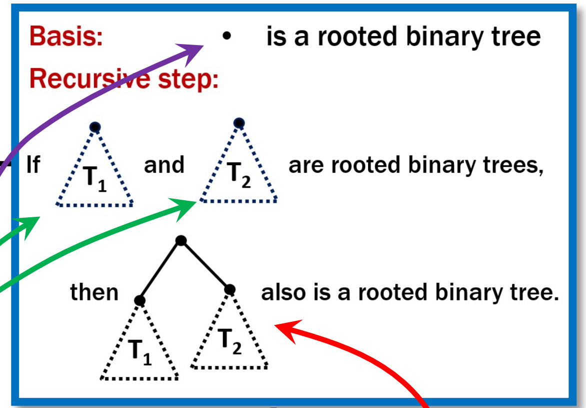
Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction



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Conclude that $\forall\, x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Structural induction follows from ordinary induction:

Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Using Structural Induction

- Let S be given by…
  - **Basis:** $6 \in S$; $15 \in S$;
  - **Recursive:** if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

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1. Let P(x) be "3 | x". We prove that P(x) is true for all x ∈ S by structural induction.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

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2. **Base Case:** 3|6 **and** 3|15 **so** P(6) **and** P(15) **are true**

Basis: $6 \in S$; $15 \in S$;

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3. **Inductive Hypothesis: Suppose that** P(x) **and** P(y) **are true for some arbitrary** x,y ∈ S
4. **Inductive Step:** **Goal: Show** P(x+y)

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

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   Since P(x) **is true,** 3|x **and so** x=3m **for some integer** m **and since** P(y) **is true,** 3|y **and so** y=3n **for some integer** n**.**
   Therefore x+y=3m+3n=3(m+n) **and thus** 3|(x+y)**.**

   Hence P(x+y) **is true.**
5. **Therefore by induction** 3|x **for all** x ∈ S.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$