CSE 311: Foundations of Computing I

Homework 6 (due November 22nd at 11:00 PM)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. However, you may use results from lecture, the theorems handout, and previous homeworks without proof.*

1. Spring Forward, Feed Back (0 points)

Approximately how much time (in minutes) did you spend on each problem of this homework? Were any problems especially difficult or especially interesting?

2. Barking Up the Strong Tree (20 points)

Let $T(n)$ denote the running time of some algorithm, where n is the size of the input. Suppose that this running time satisfies the following equations:

$$\begin{aligned} T(n) &= 8 && \text{if } n = 1 \\ T(n) &= T(\lfloor n/2 \rfloor) + 9\lfloor n/3 \rfloor + 4 && \text{for all } n > 1 \end{aligned}$$

Use strong induction to prove this *upper bound* on the running time: for all $n \geq 1$, we have $T(n) \leq 10n$.

Hint: The only facts about $\lfloor \cdot \rfloor$ that you will need are that, for all $x \in \mathbb{R}$, $\lfloor x \rfloor$ is an integer satisfying $\lfloor x \rfloor \leq x$ and, additionally, when $x \geq 1$, we also have $\lfloor x \rfloor \geq 1$.

(This running time could arise, for example, if the algorithm operates on an input of size n by, first, performing some work that takes $9\lfloor n/3 \rfloor + 4$ steps and then calling itself recursively on an input of half that size.)

3. Sum People Have All The Luck (20 points)

Let S be the set defined as follows:

Basis Step: $5 \in S$; $9 \in S$;

Recursive Step: if $x, y \in S$, then $x + y \in S$.

Show that, for every integer $n \geq 32$, it holds that $n \in S$.

Hint: Strong (not structural) induction is the right tool here since the quantifier is over n (not S).

4. Tree, Two, One (20 points)

We define the set **Trees** as follows:

Basis Elements: for any $x \in \mathbb{Z}$, Leaf$(x) \in$ **Trees**.

Recursive Step: for any $x \in \mathbb{Z}$, if $L, R \in$ **Trees**, then Branch$(x, L, R) \in$ **Trees**.

This defines a set of *binary trees* just like those defined in lecture except that we now include data in the nodes.

The following functions count the number of each type of node in the tree:

$$\begin{aligned} \text{leaves}(\text{Leaf}(x)) &= 1 && \text{for any } x \in \mathbb{Z} \\ \text{leaves}(\text{Branch}(x, L, R) &= \text{leaves}(L) + \text{leaves}(R) && \text{for any } x \in \mathbb{Z} \text{ and } L, R \in \textbf{Trees} \end{aligned}$$

$$\begin{aligned} \text{branches}(\text{Leaf}(x)) &= 0 && \text{for any } x \in \mathbb{Z} \\ \text{branches}(\text{Branch}(x, L, R) &= 1 + \text{branches}(L) + \text{branches}(R) && \text{for any } x \in \mathbb{Z} \text{ and } L, R \in \textbf{Trees} \end{aligned}$$

Prove that, for any $T \in$ **Trees**, we have leaves$(T) =$ branches$(T) + 1$.

5. Long Live the String (20 points)

Let Σ be a fixed alphabet. Prove that, for any strings $x, y, z \in \Sigma^*$, we have $x \bullet (y \bullet z) = (x \bullet y) \bullet z$. (In words, this says that "$\bullet$" is associative.) Your proof should use structural induction **on z**.

6. A Few of My Favorite Strings (15 points)

For each of the following, write a recursive definition of the set of strings satisfying the given properties. *Briefly* justify that your solution is correct.

(a) [5 Points] Binary strings that start with 1 and have odd length.

(b) [5 Points] Binary strings where every occurrence of a 0 is immediately followed by 11.

(c) [5 Points] Binary strings with an odd number of 1s.

7. Hope Strings Eternal [Online] (15 points)

For each of the following, construct regular expressions that match the given set of strings:

(a) [3 Points] Binary strings that start with 1 and have odd length.

(b) [3 Points] Binary strings where every occurrence of a 0 is immediately followed by a 11.

(c) [3 Points] Binary strings with an odd number of 1s

(d) [3 Points] Binary strings with at least two 1s.

(e) [3 Points] Binary strings with at least two 1s **or** at most one 0.

> Submit and check your answers to this question here:
>
> `https://grin.cs.washington.edu/`
>
> Think carefully about your answer to make sure it is correct before submitting. You have only 5 chances to submit a correct answer.

8. Extra Credit: Live Strong and Prosper (0 points)

Consider an infinite sequence of positions $1, 2, 3, \ldots$ and suppose we have a stone at position 1 and another stone at position 2. In each step, we choose one of the stones and move it according to the following rule: Say we decide to move the stone at position i; if the other stone is not at any of the positions $i+1, i+2, \ldots, 2i$, then it goes to $2i$, otherwise it goes to $2i+1$.

For example, in the first step, if we move the stone at position 1, it will go to 3 and if we move the stone at position 2 it will go to 4. Note: no matter how we move the stones, they will never be at the same position.

Use induction to prove that, for any given positive integer n, it is possible to move one of the stones to position n. For example, if $n = 7$ first we move the stone at position 1 to 3. Then, we move the stone at position 2 to 5 Finally, we move the stone at position 3 to 7.