CSE 311: Foundations of Computing I

Homework 4 (due Friday, October 29th at 11:00 PM)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. However, you may use results from lecture, the theorems handout, and previous homeworks without proof.*

1. Feedback, Paddy Whack, Give a Dog a Bone (0 points)

Approximately how much time (in minutes) did you spend on each problem of this homework? Were any problems especially difficult or especially interesting?

2. Fiddler On the Proof (12 points)

Consider the following claim: $(A \setminus C) \cap (B \setminus C) \subseteq (A \cap B) \setminus C$.

(a) [2 Points] What Predicate Logic statement do we need to prove to show that this "$\subseteq$" claim is true?

(b) [6 Points] Write a formal proof of the claim from (a).

(c) [4 Points] Translate the formal proof from (b) into an English proof of the original "$\subseteq$" claim.

3. Two Proofs and a Lie (16 points)

Let A, B, and C be sets. Prove or disprove (in English) the following claims.

If the claim is true, use the Meta Theorem as a *template* for your proof. If the claim is false, use the only proof strategy discussed in lecture that applies in that case (counter-example).

(a) $(B \setminus A) \cap (C \setminus A) = (B \cup C) \setminus A$

(b) $(A \cup \overline{B}) \setminus (A \cap B) = \overline{B}$

(c) $A \cup (B \setminus C) = (A \cup B) \setminus (\overline{A} \cap C)$

4. Our Finest Power (20 points)

(a) [4 Points] Disprove the claim that $\mathcal{P}(S \cap T) = \{\emptyset\} \cup (\mathcal{P}(S) \setminus \mathcal{P}(S \setminus T))$ for all sets S and T.

(b) [10 Points] Let S and T be fixed sets. Write a formal proof that $\mathcal{P}(S) \cap \mathcal{P}(T) \subseteq \mathcal{P}(S \cap T)$.

Note: Your proof should use *only* the definitions of $\mathcal{P}$, $\subseteq$, etc. and our rules of inference.

(c) [6 Points] Translate your proof from (b) into an English proof.

5. Keeping Up With the Cartesians (20 points)

(a) [4 Points] Disprove the claim that $A \times B = B \times A$ for all sets A and B.

(b) [10 Points] Let A, B, and C be fixed sets. Write a formal proof showing that $A \subseteq B$ given that B is not empty (i.e., $\exists x\, (x \in B)$) and $A \times B \subseteq B \times C$.

Note: Your proof should use *only* the definitions of $\times$, $\subseteq$, etc. and our rules of inference. Recall that the definition of "$\times$" says that $(a, b) \in A \times B$ is equivalent to $(a \in A) \wedge (b \in B)$.

(c) [6 Points] Translate your proof from (b) into an English proof.

6. Weekend At Cape Mod (16 points)

Let a, b, c, and n be integers with c and n both positive. Consider the claim that $a \equiv_n b$ holds if and only if $ca \equiv_{cn} cb$ does.

(a) [10 Points] Write a formal proof of the claim. You can infer new equations from ones already known via standard algebraic techniques by citing the rule "Algebra".

(b) [6 Points] Translate the formal proof into an English proof.

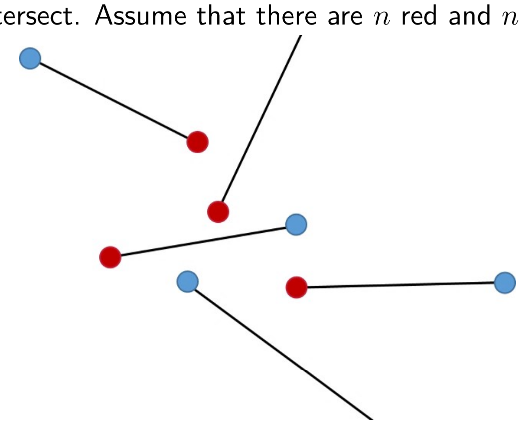
7. A Good Prime Was Had By All (16 points)

Let $p > 3$. Write an English proof that, if p is prime, then $p \equiv_6 1$ or $p \equiv_6 5$.

Hint: use one of the proof strategies discussed in lecture.

8. Extra Credit: Match Me If You Can (0 points)

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let IsMinimal(M) be the predicate that is true precisely when M is a minimal matching. Let HasCrossing(M) be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that IsMinimal(M) is true, i.e.

$$\exists M \text{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M(\text{HasCrossing}(M) \rightarrow \neg\text{IsMinimal}(M))$$

Then, use the two results above to give a proof of the statement:

$$\exists M \neg\text{HasCrossing}(M).$$