Lecture 22: FSMs w/Output, FSM Minimization & NFAs
Vending Machine

Enter 15 cents in dimes or nickels
Press S or B for a candy bar
Basic transitions on N (nickel), D (dime), B (butterfinger), S (snickers)
Adding output to states: N – Nickel, S – Snickers, B – Butterfinger
Adding additional “unexpected” transitions to cover all symbols for each state
State Minimization

• Many different FSMs (DFAs) for the same problem
• Take a given FSM and try to reduce its state set by combining states
  – Algorithm will always produce the unique minimal equivalent machine (up to renaming of states) but we won’t prove this
State Minimization Algorithm

1. Put states into groups based on their outputs (or whether they are final states or not)

2. Repeat the following until no change happens
   a. If there is a symbol $s$ so that not all states in a group $G$ agree on which group $s$ leads to, split $G$ into smaller groups based on which group the states go to on $s$

3. Finally, convert groups to states
State Minimization Example

<table>
<thead>
<tr>
<th>present state</th>
<th>next state</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>S0</td>
<td>S0 S1 S2 S3</td>
<td>1</td>
</tr>
<tr>
<td>S1</td>
<td>S0 S3 S1 S5</td>
<td>0</td>
</tr>
<tr>
<td>S2</td>
<td>S1 S3 S2 S4</td>
<td>1</td>
</tr>
<tr>
<td>S3</td>
<td>S1 S0 S4 S5</td>
<td>0</td>
</tr>
<tr>
<td>S4</td>
<td>S0 S1 S2 S5</td>
<td>1</td>
</tr>
<tr>
<td>S5</td>
<td>S1 S4 S0 S5</td>
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</table>

state transition table

Put states into groups based on their outputs (or whether they are final states or not).
State Minimization Example

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State Minimization Example

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<tr>
<th>present state</th>
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<th>2</th>
<th>3</th>
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Finally convert groups to states:

Can combine states S0-S4 and S3-S5.

In table replace all S4 with S0 and all S5 with S3.
Minimized Machine

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</table>

state transition table

n states originally
A Simpler Minimization Example
A Simpler Minimization Example

Split states into final/non-final groups

Every symbol causes the DFA to go from one group to the other so neither group needs to be split
Minimized DFA

\[ s_0 \quad 0,1 \quad s_1 \]

\[ s_3 \quad 0,1 \quad s_2 \]
Another way to look at DFAs

Definition: The label of a path in a DFA is the concatenation of all the labels on its edges in order.

Lemma: $x$ is in the language recognized by a DFA iff $x$ labels a path from the start state to some final state.
Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol—can have 0 or >1
  - Also can have edges labeled by empty string $\varepsilon$

- Definition: $x$ is in the language recognized by an NFA if and only if $x$ labels a path from the start state to some final state

![Diagram of NFA with states $s_0$, $s_1$, $s_2$, $s_3$ and transitions labeled by 0,1 and 1.]

All binary strings containing pattern...
Consider this NFA

What language does this NFA accept?

Binary strings that start with 1, followed by any number of 0s or 1s: $(1(0)(0)^*)$. 

$\{w \in \{0, 1\}^* : w$ starts with 1$\}$.
Consider This NFA

What language does this NFA accept?

\[
10(10)^* \cup 111 (0 \cup 1)^*
\]
NFA \( \varepsilon \)-moves

\[ \begin{align*}
\varepsilon & \quad \rightarrow s_0 \quad 0,1 \quad 2 \quad 2 \quad 0,1 \\
q & \quad \varepsilon \quad \rightarrow \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \\
s_0 & \quad \varepsilon \quad \rightarrow \quad t_0 \quad 2 \quad 2 \quad 1 \quad 2 \quad t_2 \\
s_1 & \quad 2 \quad \rightarrow \quad t_1 \quad 1 \quad 1 \\
t_0 & \quad 1 \quad \rightarrow \quad t_1 \quad 2 \quad 2 \\
t_1 & \quad 2 \quad \rightarrow \quad t_2 \quad 0 \\
t_2 & \quad 0 \quad \rightarrow \quad t_0 \quad 1 \\
\end{align*} \]

\( \varepsilon \)-moves with
\begin{itemize}
\item stays with 2
\item total # of 2
\item digit sum \( \equiv 0 \text{ (mod 3)} \)
\end{itemize}
Strings over \{0,1,2\} w/ even # of 2’s OR sum to 0 mod 3
Three ways of thinking about NFAs

• Outside observer: Is there a path labeled by $x$ from the start state to some final state?

• Perfect guesser: The NFA has input $x$ and whenever there is a choice of what to do it magically guesses a good one (if one exists)

• Parallel exploration: The NFA computation runs all possible computations on $x$ step-by-step at the same time in parallel
NFA for set of binary strings with a 1 in the 3\textsuperscript{rd} position from the end
NFA for set of binary strings with a 1 in the 3\textsuperscript{rd} position from the end
Compare with the smallest DFA
Parallel Exploration view of an NFA

Input string 0101100