CSE 311: Foundations of Computing

Lecture 17: Structural Induction, Regular expressions

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from the basis step and a finite number of recursive steps

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: **Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step***

Inductive Hypothesis: **Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step***

Inductive Step: **Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis**

Conclude **that $\forall\, x \in S, P(x)$**

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string w/ no chars)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$len(\varepsilon) = 0$

$len(wa) = 1 + len(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

$\#_c(\varepsilon) = 0$

$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$" .
We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** x $\in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** w $\in \Sigma^*$

Inductive Step: **Goal: Show that** P(wa) **is true for every** a $\in \Sigma$

Let a $\in \Sigma$**. Let** x $\in \Sigma^*$. **Then** len(x•wa) = len((x•w)a) **by defn of •**
= len(x•w)+1 **by defn of len**
= len(x)+len(w)+1 **by I.H.**
= len(x)+len(wa) **by defn of len**

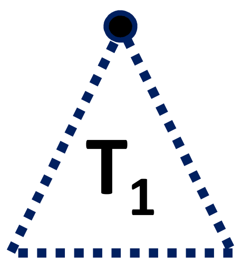
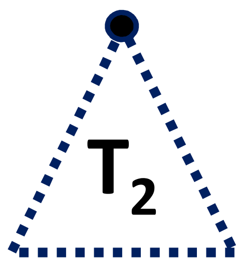
Therefore len(x•wa)= len(x)+len(wa) **for all** x $\in \Sigma^*$, **so** P(wa) **is true.**

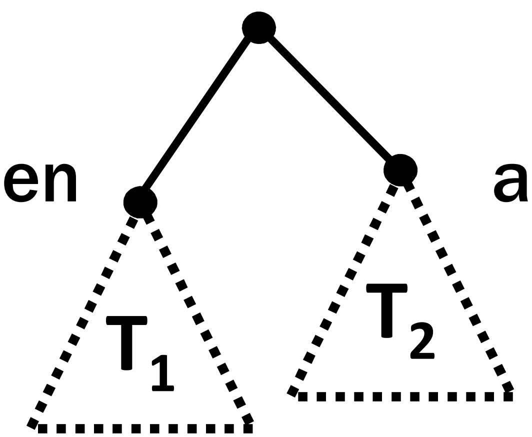
So, by induction len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Full

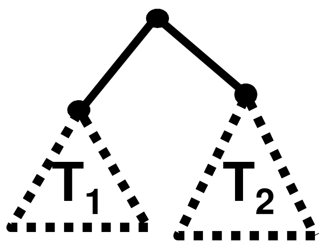
Rooted Binary Trees

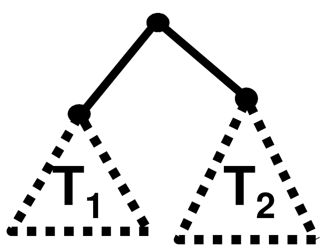
- **Basis:** • is a rooted binary tree
- **Recursive step:**

If  and  are rooted binary trees,

then  also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)
- height(•) = 0
- height (

)=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Claim: **For every rooted binary tree T,** size(**T**) ≤ $2^{\text{height}(T)+1} - 1$

Proof: 1. Let $P(T)$ be "$\text{size}(T) \le 2^{\text{height}(T)+1} - 1$"

We $P(T)$ for all rooted binary trees T by structural induction

2. Base Case: $(T = \cdot)$ height$(\cdot) = 0$

$\text{size}(\cdot) = 1 = 2 - 1 = 2^{0+1} - 1$

$= 2^{\text{height}(\cdot)+1} - 1$

$\therefore P(\cdot)$ is true

Claim: For every rooted binary tree T, size(T) ≤ $2^{\text{height(T)}+1} - 1$

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1. Let P(T) be "size(T) ≤ $2^{\text{height(T)}+1}$–1". We prove P(T) for all rooted binary trees T by structural induction.
2. **Base Case:** size(•)=1, height(•)=0 and 1=2^1–1=2^{0+1}–1 so P(•) is true.

3. Inductive Hypothesis: Assume that $P(T_1)$ and $P(T_2)$ are true for some arbitrary rooted binary trees T_1 and T_2.

4. Inductive Step:

$$\boxed{\text{Goal: Prove } P(\ )}$$

$$\text{size}(\ ) = \text{size}(T_1) + \text{size}(T_2) + 1 \quad \text{by defn}$$

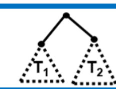
$$\leq (2^{\text{height}(T_1)+1} - 1) + (2^{\text{height}(T_2)+1} - 1) + 1 \quad \text{by I.H.}$$

$$= 2^{\text{height}(T_1)+1} + 2^{\text{height}(T_2)+1} - 1$$

$$\leq 2 \cdot \max(2^{\text{height}(T_1)+1}, 2^{\text{height}(T_2)+1}) - 1$$

$$= 2 \cdot 2^{\underbrace{\max(\text{height}(T_1), \text{height}(T_2))+1}_{=\text{height}(}} - 1$$

Claim: For every rooted binary tree T, size(T) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

1. Let P(**T**) be "size(T) $\leq 2^{\text{height(T)}+1}$–1". We prove P(**T**) for all rooted binary trees T by structural induction.
2. Base Case: size(•)=1, height(•)=0 and $1=2^1-1=2^{0+1}-1$ so P(•) is true.
3. Inductive Hypothesis: Suppose that P($\mathbf{T_1}$) and P($\mathbf{T_2}$) are true for some rooted binary trees $\mathbf{T_1}$ and $\mathbf{T_2}$.
4. Inductive Step: Goal: Prove P().

Claim: For every rooted binary tree T, size(**T**) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

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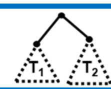
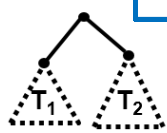
1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height}(T)+1} - 1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0 **and** $1=2^1-1=2^{0+1}-1$ **so** P(•) **is true.**
3. **Inductive Hypothesis: Suppose that** P(**T_1**) **and** P(**T_2**) **are true for some rooted binary trees T_1 and T_2.**
4. **Inductive Step:** **Goal: Prove** P().

By defn, size() $= 1+\text{size}(\mathbf{T_1})+\text{size}(\mathbf{T_2})$

by defn

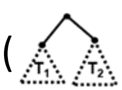
$\leq 1+2^{\text{height}(\mathbf{T_1})+1} - 1 + 2^{\text{height}(\mathbf{T_2})+1} - 1$

by IH for $\mathbf{T_1}$ and $\mathbf{T_2}$

$= 2^{\text{height}(\mathbf{T_1})+1} + 2^{\text{height}(\mathbf{T_2})+1} - 1$

$\leq 2(2^{\max(\text{height}(\mathbf{T_1}),\text{height}(\mathbf{T_2}))+1}) - 1$

by defn

= 2(2^height()) - 1 = 2^height()+1 - 1

which is what we wanted to show.

5. **So, the** P(T) **is true for all rooted bin. trees by structural induction.**

Languages: Sets of Strings

- Sets of strings that satisfy special properties are called *languages*. Examples:
  - English sentences
  - Syntactically correct Java/C/C++ programs
  - $\Sigma^* =$ All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Legal variable names. keywords in Java/C/C++
  - Binary strings with an equal # of 0's and 1's

Regular Expressions

Regular expressions over Σ

- **Basis:**

  $\varnothing$, ε are regular expressions

  a is a regular expression for any $a \in \Sigma$

- **Recursive step:**
  - If **A** and **B** are regular expressions then so are:

    (A $\cup$ B)

    (AB)

    A*

Each Regular Expression is a "pattern"

ε matches the **empty string**

a matches the one character string *a*

($A \cup B$) matches all strings that either **A** matches or **B** matches (or both)

(AB) matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another

Examples

001*

{00, 001, 0011, 00111, . . . }

0*1*

All binary strings that don't have a 0 after a 1.

Any # of 0's followed by any # of 1's

Examples

001*

{00, 001, 0011, 00111, ...}

0*1*

Any number of 0's followed by any number of 1's

Examples

$(0 \cup 1)\ 0\ (0 \cup 1)\ 0$

$(0^*1^*)^*$

Examples

$(0 \cup 1)\ 0\ (0 \cup 1)\ 0$

{0000, 0010, 1000, 1010}

$(0^*1^*)^*$

All binary strings

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

Binary strings that contain "0110"

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Binary strings that begin with pairs of characters followed by "01010" or "10001"

Regular Expressions in Practice

- Used to define the "tokens": e.g., legal variable names, keywords in programming languages and compilers
- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of PHP
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

- Pattern p = Pattern.compile("a*b");
- Matcher m = p.matcher("aaaaab");
- boolean b = m.matches();

`[01]`	a 0 or a 1	`^` start of string		`$` end of string			
`[0–9]`	any single digit	`\.`	period	`\,`	comma	`\–`	minus
`.`	any single character						

ab	a followed by b	(**AB**)
(a|b)	a or b	(**A** $\cup$ **B**)
a**?**	zero or one of a	(**A** $\cup$ ε)
a*****	zero or more of a	**A***
a**+**	one or more of a	**AA***

- e.g. `^[\–+]?[0–9]*(\.|\,)?[0–9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)