Recursive definition of set $S$

- **Basis Step:** $0 \in S$
- **Recursive Step:** If $x \in S$, then $x + 2 \in S$
- **Exclusion Rule:** Every element in $S$ follows from the basis step and a finite number of recursive steps.

We need the exclusion rule because otherwise $S = \mathbb{N}$ would satisfy the other two parts. However, we won’t always write it down on these slides.

$S = \{\text{even natural numbers}\}$
Last class: Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: If $x, y \in S$, then $x + y \in S$

\[ S = \{6, 12, 15, 18, 21, \ldots \} \]

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for any $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

\[ S = \{ \text{plane in } \mathbb{R}^3 \text{ spanned by } [1,1,0] \text{ and } [0,1,1] \} \]

Number of form $3^n$ for $n \geq 0$:

Basis: $1 \in S$
Recursive: If $x \in S$, then $3x \in S$. 
Recursive Definitions of Sets: General Form

Recursive definition

– *Basis step*: Some specific elements are in $S$

– *Recursive step*: Given some existing named elements in $S$ some new objects constructed from these named elements are also in $S$.

– *Exclusion rule*: Every element in $S$ follows from the basis step and a finite number of recursive steps
Strings

• An alphabet $\Sigma$ is any finite set of characters

• The set $\Sigma^*$ of strings over the alphabet $\Sigma$ is defined by
  – Basis: $\varepsilon \in \Sigma^*$ ($\varepsilon$ is the empty string with no characters)
  – Recursive: if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$
Palindromes

Palindromes are strings that are the same backwards and forwards

**Basis:**

\( \varepsilon \) is a palindrome and any \( a \in \Sigma \) is a palindrome

**Recursive step:**

If \( p \) is a palindrome then \( apa \) is a palindrome for every \( a \in \Sigma \)
All Binary Strings with no 1’s before 0’s
All Binary Strings with no 1’s before 0’s

Basis:
\[ \varepsilon \in S \]

Recursive:
\[ \text{If } x \in S, \text{ then } 0x \in S \]
\[ \text{If } x \in S, \text{ then } x1 \in S \]
Functions on Recursively Defined Sets (on $\Sigma^*$)

Length:
$$\text{len}(\varepsilon) = 0$$
$$\text{len}(wa) = 1 + \text{len}(w) \text{ for } w \in \Sigma^*, \ a \in \Sigma$$

Reversal:
$$\varepsilon^R = \varepsilon$$
$$(wa)^R = aw^R \text{ for } w \in \Sigma^*, \ a \in \Sigma$$

Concatenation:
$$x \cdot \varepsilon = x \text{ for } x \in \Sigma^*$$
$$x \cdot wa = (x \cdot w)a \text{ for } x, w \in \Sigma^*, \ a \in \Sigma$$

Number of $c$’s in a string:
$$\#_c(\varepsilon) = 0$$
$$\#_c(wa) = \#_c(w) \text{ for } w \in \Sigma^*, \ a \in \Sigma, \ a \neq c$$
$$\#_c(wc) = \#_c(w) + 1 \text{ for } w \in \Sigma^*$$
Rooted Binary Trees

- **Basis:** is a rooted binary tree
- **Recursive step:**

If $T_1$ and $T_2$ are rooted binary trees, then $\begin{array}{c} T_1 \\ \text{and} \\ T_2 \end{array}$ also is a rooted binary tree.
Defining Functions on Rooted Binary Trees

- \( \text{size}(\bullet) = 1 \)

- \( \text{size} ( \ T_1 \ T_2 \ ) = 1 + \text{size}(T_1) + \text{size}(T_2) \)

- \( \text{height}(\bullet) = 0 \)

- \( \text{height} ( \ T_1 \ T_2 \ )=1 + \max\{\text{height}(T_1), \text{height}(T_2)\} \)
Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all specific elements $u$ of $S$ mentioned in the *Basis step*

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

**Inductive Step:** Prove that $P(w)$ holds for each of the new elements $w$ constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

**Conclude** that $\forall x \in S, P(x)$
Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all **specific elements** $u$ of $S$ mentioned in the **Basis step**

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of each of the **existing named elements** mentioned in the **Recursive step**

**Inductive Step:** Prove that $P(w)$ holds for each of the **new elements** $w$ constructed in the **Recursive step** using the named elements mentioned in the **Inductive Hypothesis**

**Conclude** that $\forall x \in S, P(x)$
Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$
- **Basis:** $0 \in \mathbb{N}$
- **Recursive step:** If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be “for all $x \in S$ that can be constructed in at most $n$ recursive steps, $P(x)$ is true.”
Using Structural Induction

• Let $S$ be given by...
  – **Basis:** $6 \in S; \quad 15 \in S;$
  – **Recursive:** if $x, y \in S$ then $x + y \in S.$

**Claim:** Every element of $S$ is divisible by 3.
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true

Basis: $6 \in S; \ 15 \in S$

Recursive: if $x, y \in S$ then $x + y \in S$
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

4. Inductive Step: Goal: Show $P(x+y)$

Basis: $6 \in S$; $15 \in S$;
Recursive: if $x, y \in S$ then $x + y \in S$
**Claim:** Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. **Base Case:** $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

3. **Inductive Hypothesis:** Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

4. **Inductive Step:** **Goal:** Show $P(x+y)$

   Since $P(x)$ is true, $3 \mid x$ and so $x = 3m$ for some integer $m$ and since $P(y)$ is true, $3 \mid y$ and so $y = 3n$ for some integer $n$.

   Therefore $x + y = 3m + 3n = 3(m+n)$ and thus $3 \mid (x+y)$.

   Hence $P(x+y)$ is true.

5. Therefore by induction $3 \mid x$ for all $x \in S$.

**Basis:** $6 \in S$; $15 \in S$;

**Recursive:** if $x, y \in S$ then $x + y \in S$
Claim: \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.

We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.
Claim: \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.

We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

**Base Case:** \( y = \varepsilon \). For any \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). Therefore \( P(\varepsilon) \) is true.
**Claim:** \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

**Base Case:** \( y = \varepsilon \). For any \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). Therefore \( P(\varepsilon) \) is true.

**Inductive Hypothesis:** Assume that \( P(w) \) is true for some arbitrary \( w \in \Sigma^* \).

**Inductive Step:** *Goal: Show that \( P(wa) \) is true for every \( a \in \Sigma \)*
**Claim:** \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.

We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

**Base Case:** \( y = \varepsilon \). For any \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). Therefore \( P(\varepsilon) \) is true.

**Inductive Hypothesis:** Assume that \( P(w) \) is true for some arbitrary \( w \in \Sigma^* \).

**Inductive Step:** Goal: Show that \( P(wa) \) is true for every \( a \in \Sigma \)

Let \( a \in \Sigma \). Let \( x \in \Sigma^* \). Then \( \text{len}(x \cdot wa) = \text{len}((x \cdot w)a) \) by defn of \( \cdot \)

\[
= \text{len}(x \cdot w) + 1 \quad \text{by defn of \( \cdot \)}
= \text{len}(x) + \text{len}(w) + 1 \quad \text{by I.H.}
= \text{len}(x) + \text{len}(wa) \quad \text{by defn of \( \text{len} \)}
\]

Therefore \( \text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa) \) for all \( x \in \Sigma^* \), so \( P(wa) \) is true.

So, by induction \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)