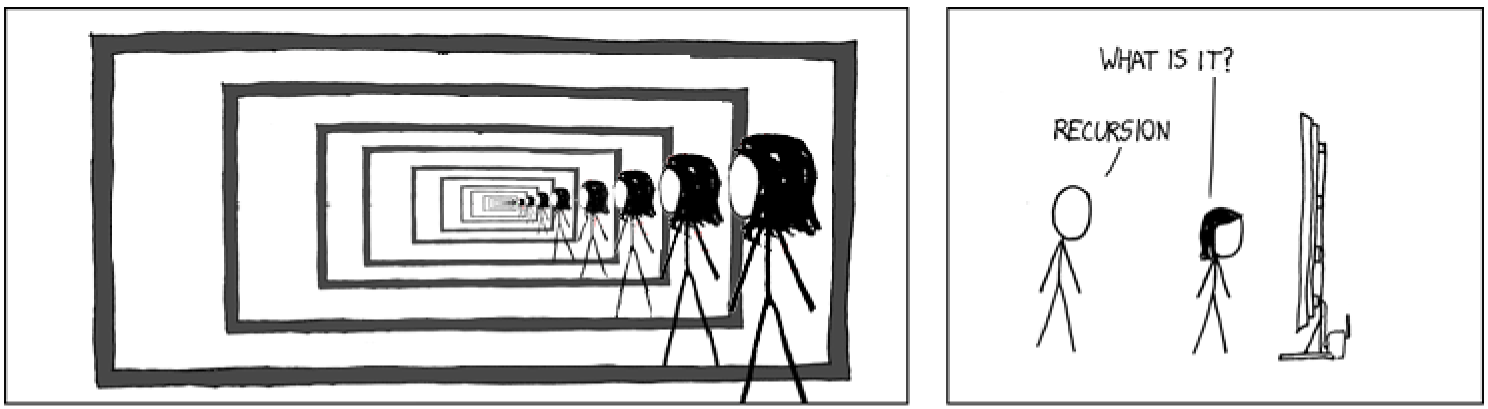
CSE 311: Foundations of Computing

Lecture 16: Recursively Defined Sets & Structural Induction

instead



- HW 5 solutions in section tomorrow
- Review Session 5:00–6:30 pm Sieg 134 Tomorrow: bring Questions!
- Midterm Friday: Bring writing instruments (dark) - DON'T ERASE

Last class: Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step: If** $x \in S$**, then** $x + 2 \in S$
- **Exclusion Rule: Every element in** S **follows from the basis step and a finite number of recursive steps.**

S={even natural numbers}

We need the exclusion rule because otherwise $S=\mathbb{N}$ **would satisfy the other two parts. However, we won't always write it down on these slides.**

Last class: Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: **If** $x, y \in S$, **then** $x+y \in S$

S={6,12,15,18,21,...}

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: **If** $[x, y, z] \in S$, **then** $[\alpha x, \alpha y, \alpha z] \in S$ **for any** $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ **and** $[x_2, y_2, z_2] \in S$, **then**
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

S={**plane in** $\mathbb{R}^3$ **spanned by [1,1,0] and [0,1,1]**}

Number of form 3^n **for** $n \geq 0$**:**
Basis: $1 \in S$
Recursive: If $x \in S$, **then** $3x \in S$.

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from the basis step and a finite number of recursive steps

Strings

Σ^*

- An *alphabet* Σ is any finite set of characters

  e.g. $\Sigma = \{0,1\}$ $\Sigma = \{a, b, \ldots, z\}$ Σ = ASCII Σ = UNICODE

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string w/ no chars)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

$\{0,1\}^*$ set of all binary strings

ε → 0 → 00, 01; → 1 → 10, 11

finite length

Palindromes

Palindromes are strings that are the same backwards and forwards

rotor
deed

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

t oto rotor

ε ee deed

All Binary Strings with no 1's before 0's

Basis: $\varepsilon, 0, 1 \in S$

Recursive: If $p \in S$ then $0p \in S$ and $p1 \in S$ ✓

Basis: $\varepsilon \in S$

Recursive: If $p \in S$ then $0p \in S$ and $p1 \in S$ ✓

Basis: $\varepsilon \in S$

Recursive: If $p \in S$ then If p contains 1 then $p1 \in S$; If p not contains 1 then $p0$ and $p1 \in S$

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, **then** $0x \in S$

If $x \in S$, **then** $x1 \in S$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

$(110)^R = 0(11)^R$
$= 01 1^R$
$= 011\varepsilon^R$
$= 011\varepsilon = 011$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$, $w \in \Sigma^*$

Number of c's in a string:

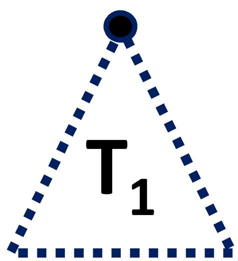
$\#_c(\varepsilon) = 0$

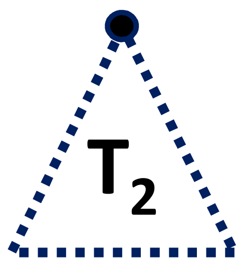
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

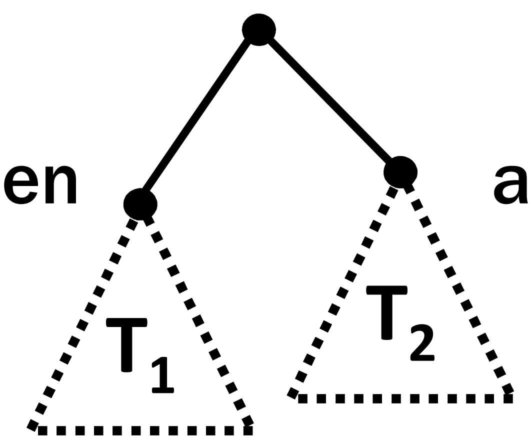
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

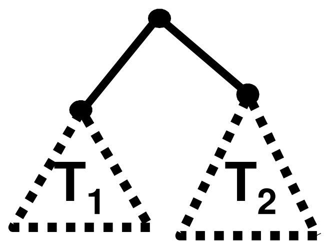
and 

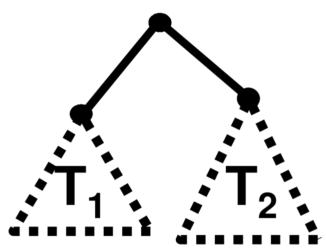
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)
- height(•) = 0
- height (

)=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

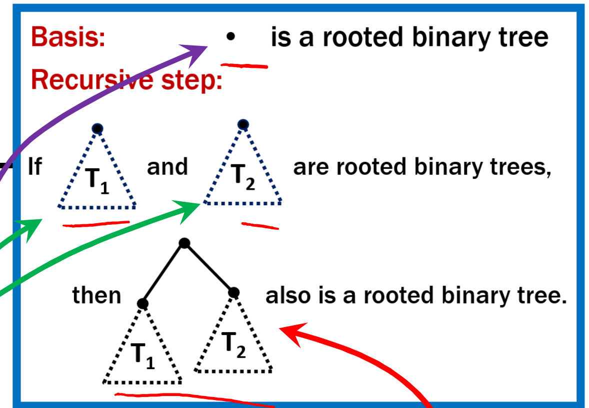
Base Case: **Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step***

Inductive Hypothesis: **Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step***

Inductive Step: **Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis**

Conclude **that $\forall\, x \in S, P(x)$**

Structural Induction



How to prove $\forall\, x \in S, P(x)$ is true:

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Conclude that $\forall\, x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: **If** $k \in \mathbb{N}$ **then** $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ **to be "for all** $x \in S$ **that can be constructed in at most** n **recursive steps,** $P(x)$ **is true."**

Using Structural Induction

- Let S be given by…
  - Basis: $6 \in S$; $15 \in S$;
  - Recursive: if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

1. $P(x)$ is "x is divisible by 3". We prove $P(x)$ for all $x \in S$ by induction
2. Base Case: $3 | 6$ and $3 | 15$ so $P(6)$, $P(15)$ are true
3. IH: Assume that $3 | x$ and $3 | y$ for some arbitrary $x, y \in S$
4. I.S. By IH $x = 3k$ and $y = 3l$ for some integers k, l

$\therefore x + y = 3k + 3l = 3(k+l)$

$\therefore 3 | (x+y)$ $\therefore P(x+y)$

$\therefore$ By induction every elt of S is divisible by 3 ∎

Claim: Every element of S is divisible by 3.

1. Let P(x) be "3|x". We prove that P(x) is true for all x ∈ S by structural induction.
2. **Base Case:** 3|6 and 3|15 so P(6) and P(15) are true

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. **Let** P(x) **be** "3|x". **We prove that** P(x) **is true for all** $x \in S$ **by structural induction.**
2. **Base Case:** 3|6 **and** 3|15 **so** P(6) **and** P(15) **are true**
3. **Inductive Hypothesis: Suppose that** P(x) **and** P(y) **are true for some arbitrary** $x,y \in S$
4. **Inductive Step:** **Goal: Show** P(x+y)

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "3|x". We prove that P(x) is true for all x ∈ S by structural induction.
2. Base Case: 3|6 and 3|15 so P(6) and P(15) are true
3. Inductive Hypothesis: Suppose that P(x) and P(y) are true for some arbitrary x,y ∈ S
4. Inductive Step: Goal: Show P(x+y)

   Since P(x) is true, 3|x and so x=3m for some integer m and since P(y) is true, 3|y and so y=3n for some integer n. Therefore x+y=3m+3n=3(m+n) and thus 3|(x+y).

   Hence P(x+y) is true.
5. Therefore by induction 3|x for all x ∈ S.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: len(x•y) = len(x) + len(y) for all x,y ∈ Σ*

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x ∈ Σ* ".
We prove P(y) **for all** y ∈ Σ* **by structural induction.**

Base Case: y = ε   len(x·ε) = len(x) by defn of •
= len(x) + 0
= len(x) + len(ε) by defn of len
∴ P(ε) is true

Ind. Hypoth:

Claim: len(x•y) = len(x) + len(y) **for all** $x,y \in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all $x \in \Sigma^*$" .
We prove P(y) **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** $x \in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Ind Hyp: Suppose that P(w) is true for some arbitrary $w \in \Sigma^*$ (len(x•w) = len(x)+len(w) for all $x \in \Sigma^*$)

Ind Step: Goal: Show P(wa) is true for all $a \in \Sigma$

len(x•wa) = len((x•w)a) by defn of •
= 1+ len(x•w) by defn of len
= 1+ len(x) + len(w) by IH
= len(x) + 1 + len(w)
= len(x) + len(wa) by defn of len

x is arbitrary

∴ P(wa) ✓

If $w \in \Sigma^*$, $a \in \Sigma$ so $wa \in \Sigma^*$

Claim: $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x,y \in \Sigma^*$

Let $P(y)$ **be** "$\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$".
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, $\text{len}(x \bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ **since** $\text{len}(\varepsilon)=0$. **Therefore** $P(\varepsilon)$ **is true**

Inductive Hypothesis: **Assume that** $P(w)$ **is true for some arbitrary** $w \in \Sigma^*$

Inductive Step: **Goal: Show that** $P(wa)$ **is true for every** $a \in \Sigma$

Claim: len(x•y) = len(x) + len(y) **for all** $x,y \in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all $x \in \Sigma^*$" .
We prove P(y) **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** $x \in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** $w \in \Sigma^*$

Inductive Step: **Goal: Show that** P(wa) **is true for every** $a \in \Sigma$

Let $a \in \Sigma$**. Let** $x \in \Sigma^*$. **Then** len(x•wa) = len((x•w)a) **by defn of •**
= len(x•w)+1 **by defn of len**
= len(x)+len(w)+1 **by I.H.**
= len(x)+len(wa) **by defn of len**

Therefore len(x•wa)= len(x)+len(wa) **for all** $x \in \Sigma^*$**, so** P(wa) **is true.**

So, by induction len(x•y) = len(x) + len(y) **for all** $x,y \in \Sigma^*$