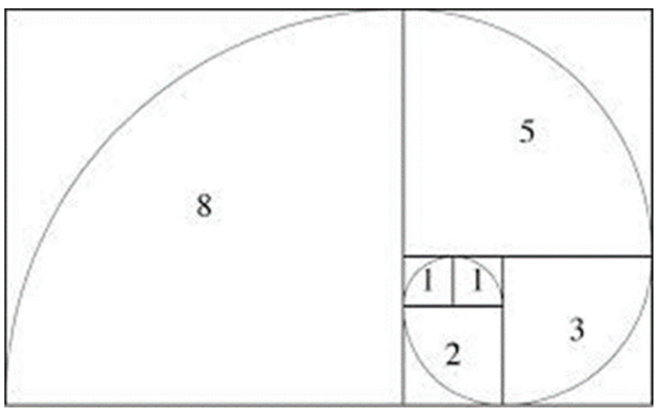
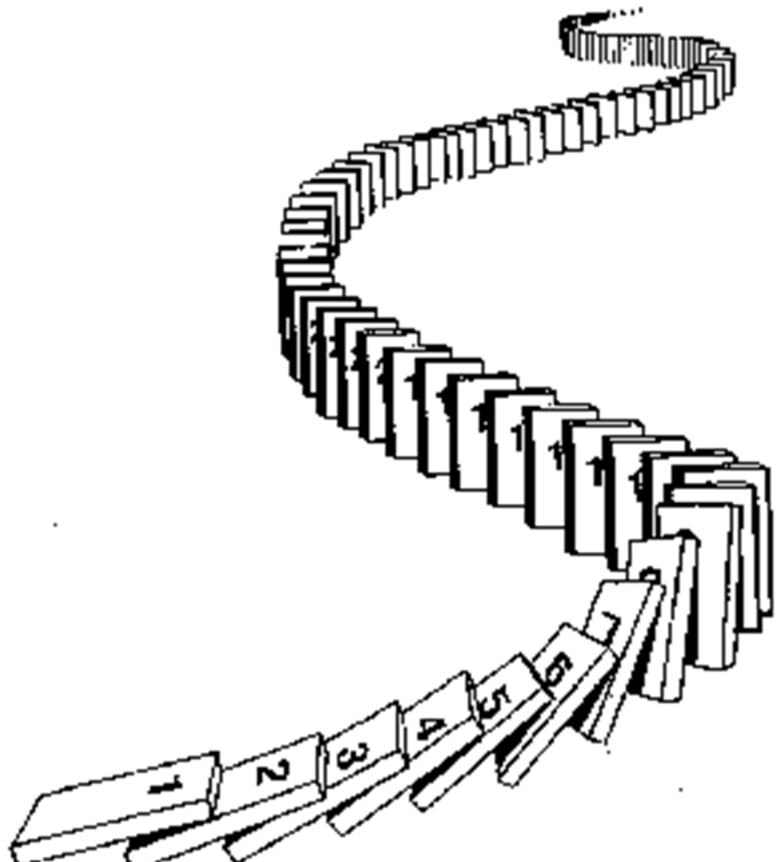
CSE 311: Foundations of Computing

Lecture 15: Recursion & Strong Induction
Applications: Fibonacci & Euclid





More Recursive Definitions

Suppose that $h\colon \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

$\sum_{i=0}^{0} h(i) = h(0)$

$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^{n} h(i)$ **for** $n \geq 0$

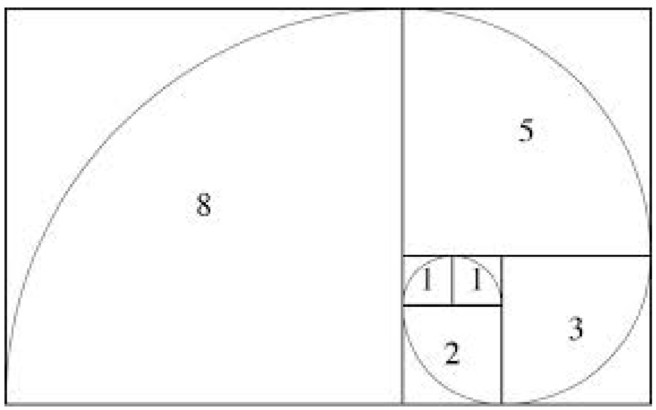
There is also product notation:

$\prod_{i=0}^{0} h(i) = h(0)$

$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^{n} h(i)$ **for** $n \geq 0$

Fibonacci Numbers

$f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ **for all** $n \geq 2$



Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ *is true for every integer* j *from* b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, P(j) **is true for every integer** j **from** 0 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} < 2^{k+1}}$

$$\boldsymbol{f_0 = 0 \quad f_1 = 1}$$
$$\boldsymbol{f_n = f_{n-1} + f_{n-2}} \text{ for all } \boldsymbol{n \geq 2}$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, P(j) is true for every integer j from 0 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$

   <u>Case k+1 = 1</u>:

   <u>Case k+1 ≥ 2</u>:

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, P(j) is true for every integer j from 0 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$

   Case $k+1 = 1$: Then $f_1 = 1 < 2 = 2^1$ so P(k+1) is true here.

   Case $k+1 \geq 2$: Then $f_{k+1} = f_k + f_{k-1}$ by definition

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) is true in this case.

   These are the only cases so P(k+1) follows.

5. Therefore by strong induction,

   $f_n < 2^n$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2\,-\,1}$ for all $n \geq 2$

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$

$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be** "$f_n \geq 2^{n/2-1}$". **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2 - 1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: $\boxed{\text{Goal: Show P(k+1); that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

No need for cases for the definition here:

$f_{k+1} = f_k + f_{k-1}$ since $k+1 \geq 2$

Now just want to apply the IH to get P(k) and P(k-1):

Problem: Though we can get P(k) since $k \geq 2$,

k-1 may only be 1 so we can't conclude P(k-1)

Solution: Separate cases for when k-1=1 (or k+1=3).

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$

$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

   <u>Case k = 2</u>:

   <u>Case k ≥ 3</u>:

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   <u>Case k = 2</u>: **Then** $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   <u>Case k ≥ 3</u>:

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

   <u>Case k = 2</u>: Then $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   <u>Case k ≥ 3</u>: $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) is true in both cases.
5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

An alternative Strong Inductive Proof Layout

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Cases:" Prove $P(b), P(b+1), \dots, P(c)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq c$,

   $P(j)$ is true for every integer j from b to k"

   These are different
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \dots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Alternative II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Cases (n=2,3): $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true. Also $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ so P(3) is true
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 3$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

   Now $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) is true.
5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$$\boldsymbol{f_0 = 0 \quad f_1 = 1}$$
$$\boldsymbol{f_n = f_{n-1} + f_{n-2}} \text{ for all } \boldsymbol{n} \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   <u>Case $k = 2$</u>: Then $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   <u>Case $k \geq 3$</u>: $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) is true in both cases.
5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$$
\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\dots \\
r_3 &= q_2 r_2 + r_1 \\
r_2 &= q_1 r_1
\end{aligned}
$$

For all $k \geq 2$, $r_{k-1} = r_{k+1} \bmod r_k$

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K's **by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k's. **After that reduction,** $r_k = f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

We go by strong induction on n**.**

Let P(n) **be** "gcd(a,b) **with** $a \geq b>0$ **takes** n **steps** $\rightarrow a \geq f_{n+1}$" **for all** $n \geq 1$**.**

<u>**Base Case**</u>: n=1 **If Euclid's Algorithm on** a, b **with** $a \geq b > 0$ **takes** 1 **step, then** $a=q_1b$ **for some** q_1 **and** $a \geq b \geq 1=f_2$ **and** P(1) **holds**

<u>**Induction Hypothesis**</u>: **Suppose that for some integer** $k \geq 1$, P(j) **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>**Inductive Step**</u>: **We want to show: if** gcd(a,b) **with** $a \geq b > 0$ **takes** k+1 **steps, then** $a \geq f_{k+2.}$

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: Suppose that for some integer $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: We want to show: if $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2.}$

Now if $k+1=2$**, then Euclid's algorithm on** a **and** b **can be written as**

$$a = q_2 b + r_1$$
$$b = q_1 r_1$$

and $r_1 > 0$.

Also, since $a \geq b>0$ **we must have** $q_2 \geq 1$ and $b \geq 1$.

So $a = q_2 b + r_1 \geq b + r_1 \geq 1+1 = 2 = f_3 = f_{k+2}$ as required.

Running time of Euclid's algorithm

Induction Hypothesis: Suppose that for some integer $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: We want to show: if $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2.}$

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1} b + r_k$$
$$b = q_k \; r_k + r_{k-1}$$
$$r_k = q_{k-1} r_{k-1} + r_{k-2}$$

and there are k-2 **more steps after this. Note that this means that the** $\gcd(b, r_k)$ **takes** k **steps and** $\gcd(r_k, r_{k-1})$ **takes** k-1 **steps and** $b > r_k > r_{k-1}$. **So since** $k, k-1 \geq 1$ **by the IH we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$.

Also, since $a \geq b$ **we must have** $q_{k+1} \geq 1$.

So $a = q_{k+1} b + r_k \geq b + r_k \geq f_{k+1} + f_k = f_{k+2}$ **as required.** ∎

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2\,-\,1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid's Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

so $(n-1)/2 \leq \log_2 a$ or $n \leq 1 + 2\log_2 a$
i.e., # of steps $\leq$ twice the # of bits in a.

Recursive Definition of Sets

Recursive Definition

- Basis Step: $0 \in S$
- Recursive Step: If $x \in S$, then $x + 2 \in S$
- Exclusion Rule: Every element in S follows from basis steps and a finite number of recursive steps.

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: If $x,y \in S$, then $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for all $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Powers of 3:

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: If $x, y \in S$, then $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for all $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Powers of 3:
Basis: $1 \in S$
Recursive: If $x \in S$, then $3x \in S$.

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

Strings

- An *alphabet* Σ is any finite set of characters
- The set Σ^* of *strings* over the alphabet Σ is defined by
  - Basis: $\varepsilon \in \Sigma$ (ε is the empty string)
  - Recursive: if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1's before 0's

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, **then** $0x \in S$

If $x \in S$, **then** $x1 \in S$

Function Definitions on Recursively Defined Sets

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$