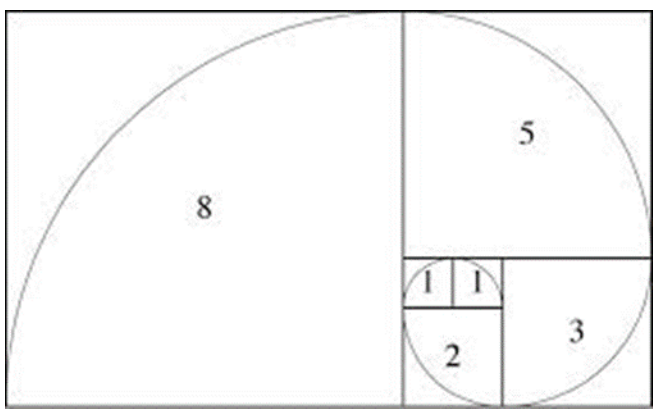
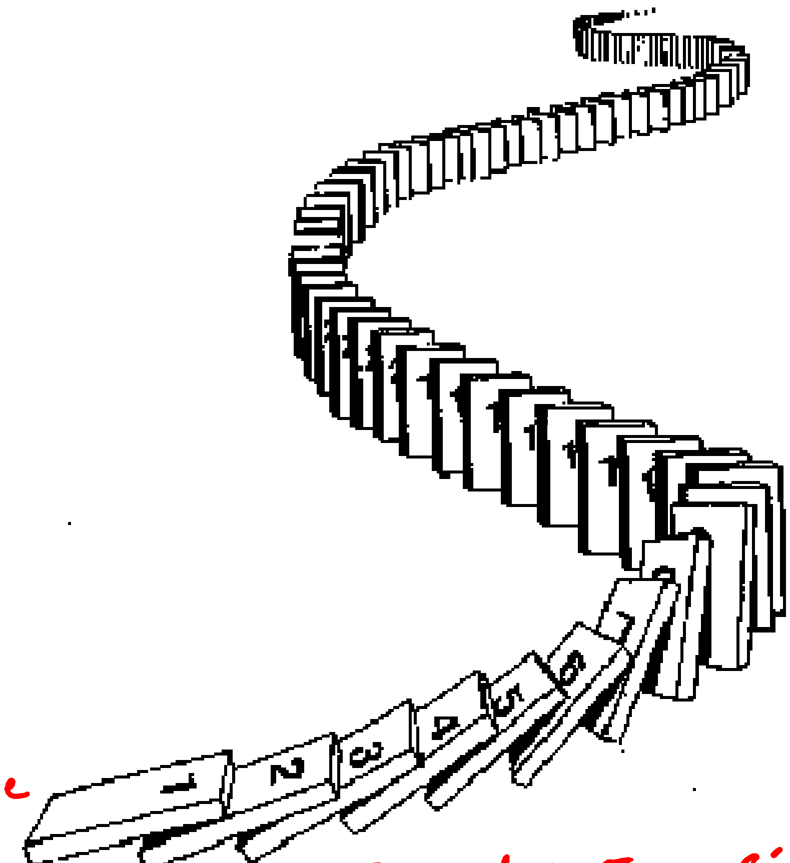
CSE 311: Foundations of Computing

Lecture 15: Recursion & Strong Induction
Applications: Fibonacci & Euclid





Midterm Friday
Practice Sol'n on web

Shorter office hour today for me
2:30-3:20

Review Session Thursday 5:00-6:30pm
Sieg 134

More Recursive Definitions

$\sum_{i=0}^{n} h(i) = h(0)+h(1)+\cdots+h(n)$

Suppose that $h \colon \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

$\sum_{i=0}^{0} h(i) = h(0)$

$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^{n} h(i)$ **for** $n \geq 0$

There is also product notation: $\prod_{i=0}^{n} h(i)$

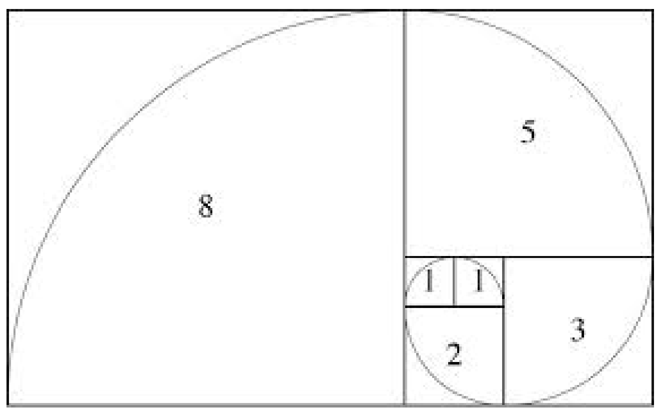
$\prod_{i=0}^{0} h(i) = h(0)$

$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^{n} h(i)$ **for** $n \geq 0$

$= h(0)\cdot h(1)\cdot h(2)\cdots h(n)$

Fibonacci Numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$



0 1 1 2 3 5 8 13 21 34 55 . .

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   $P(b)$ - - - $P(k)$

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let $P(n)$ be "$f_n < 2^n$". We will prove $P(n)$ is true for all $n \geq 0$ by strong induction.

2. Base Case: $n=0$.

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.

3. I.H. For some integer $k \geq 0$, P(j) is true for every integer j between 0 and k

4. I.S. Goal: Prove P(k+1)

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, P(j) is true for every integer j from 0 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} < 2^{k+1}}$

$f_{k+1} = \begin{cases} f_1 = 1 & \text{if } k=0 \quad k+1=1 \\ f_k + f_{k-1} & k>0 \quad k+1 \geq 2 \end{cases}$

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, P(j) is true for every integer j from 0 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} < 2^{k+1}}$

   Case k+1 = 1: $f_{k+1} = 1 < 2 = 2^{k+1}$ ∴ P(k+1) is true

   Case k+1 ≥ 2: $f_{k+1} = f_k + f_{k-1}$ $\quad k, k-1 \geq 0$

   $< 2^k + 2^{k-1}$ by IH

   $< 2^k + 2^k = 2^{k+1}$ ∴ P(k+1) is true

   P(k+1) is true in both cases

5. By strong induction $f_n < 2^n$ for all $n \geq 0$

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, P(j) **is true for every integer** j **from** 0 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} < 2^{k+1}}$

   <u>Case k+1 = 1</u>: **Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   <u>Case k+1 ≥ 2</u>: **Then** $f_{k+1} = f_k + f_{k-1}$ **by definition**

   $< 2^k + 2^{k-1}$ **by the IH since** $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) **is true in this case.**

   These are the only cases so P(k+1) **follows.**

5. **Therefore by strong induction,**

   $f_n < 2^n$ **for all integers** $n \geq 0$.

$$\boxed{\begin{array}{l} \boldsymbol{f_0 = 0 \quad f_1 = 1} \\ \boldsymbol{f_n = f_{n-1} + f_{n-2}} \ \ \textbf{for all } \boldsymbol{n \geq 2} \end{array}}$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let $P(n)$ be $f_n \geq 2^{n/2-1}$. We prove $P(n)$ for all $n \geq 2$ by strong induction

2. Base Case: ($n=2$) $f_2 = 1 \geq 1 = 2^{2/2-1}$

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
3. I.H. Assume for some $k \geq 2$ that for all j between 2 and k $P(j)$ is true
4. I.S. $\boxed{\text{Goal: Prove } P(k+1) \text{ "} f_{k+1} \geq 2^{(k+1)/2-1}\text{"}}$

$f_{k+1} = f_k + f_{k-1}$ since $k \geq 2$

$\boxed{k=2}$ $f_{k+1} = 2 \geq \sqrt{2} = 2^{1/2} = 2^{(k+1)/2-1}$ ✓

$\boxed{k \geq 3}$ $f_{k+1} = f_k + f_{k-1}$

$\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by IH

$\geq 2 \cdot 2^{(k-1)/2-1}$

$= 2^{((k-1)/2+1)-1} = 2^{((k+1)/2)-1}$ ✓

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be** "$f_n \geq 2^{n/2 - 1}$". **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} \geq 2^{(k+1)/2 - 1}$

No need for cases for the definition here:

$f_{k+1} = f_k + f_{k-1}$ **since** $k+1 \geq 2$

Now just want to apply the IH to get P(k) **and** P(k-1)**:**

Problem: Though we can get P(k) **since** $k \geq 2$,

k-1 **may only be** 1 **so we can't conclude** P(k-1)

Solution: Separate cases for when k-1=1 **(or** k+1=3**).**

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2 - 1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2 - 1}$

   <u>Case k = 2</u>:

   <u>Case k ≥ 3</u>:

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} \geq 2^{(k+1)/2-1}$

   <u>Case k = 2</u>: **Then** $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   <u>Case k ≥ 3</u>:

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

   Case k = 2: Then $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   Case k ≥ 3: $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) is true in both cases.
5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

An alternative Strong Inductive Proof Layout

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Cases:" Prove $P(b), P(b+1), \ldots, P(c)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq c$,

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

These are different

Alternative II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Cases (n=2,3): $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true. Also $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ so P(3) is true
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 3$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: $\boxed{\text{Goal: Show P(k+1); that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   Now $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) is true.

5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   Case k = 2: Then $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   Case k ≥ 3: $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) is true in both cases.
5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$$r_{n+1} = q_n r_n + r_{n-1}$$
$$r_n = q_{n-1} r_{n-1} + r_{n-2}$$
$$\ldots$$
$$r_3 = q_2 r_2 + r_1$$
$$r_2 = q_1 r_1 + \underset{0}{\underset{\shortparallel}{r_0}}$$

For all $k \geq 2$, $r_{k-1} = r_{k+1} \bmod r_k$

$$r_2 = \underset{1}{\underset{\shortparallel}{q_1}} r_1 + \underset{0}{\underset{\shortparallel}{r_0}}$$

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K's **by** 1 **and replace** r_1 **by** 1 , **we can only reduce the** r_k's.

$r_2 = r_1 + r_0$

After that reduction, r_k=f_k **for every** k.

$r_1 = 1$ $\quad$ q_i's $= 1$

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n. fails

Let P(n) **be** "gcd(a,b) **with** a ≥ b>0 **takes** n **steps** → $a \geq f_{n+1}$" **for all** n ≥ 1.

Base Case: n=1 **If Euclid's Algorithm on** a, b **with** a ≥ b > 0 **takes** 1 **step, then** $a=q_1 b$ **for some** q_1 **and** $a \geq b \geq 1 = f_2$ **and** P(1) **holds**

Induction Hypothesis: **Suppose that for some integer** k ≥ 1, P(j) **is true for all integers** j **s.t.** 1 ≤ j ≤ k

Inductive Step: **We want to show: if** gcd(a,b) **with** a ≥ b > 0 **takes** k+1 **steps, then** $a \geq f_{k+2}$.

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: **We want to show: if** $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Now if $k+1=2$, **then Euclid's algorithm on** a **and** b **can be written as**

$$a = q_2 b + r_1$$
$$b = q_1 r_1$$

and $r_1 > 0$.

Also, since $a \geq b>0$ **we must have** $q_2 \geq 1$ and $b \geq 1$.

So $a = q_2 b + r_1 \geq b + r_1 \geq 1+1 = 2 = f_3 = f_{k+2}$ as required.

n	0	1	2	3
f_n	0	1	1	2

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: **We want to show: if** $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1}b + r_k$$
$$b = q_k\ r_k + r_{k-1}$$
$$r_k = q_{k-1}r_{k-1} + r_{k-2}$$

and there are $k-2$ **more steps after this. Note that this means that the** $\gcd(b, r_k)$ **takes** k **steps and** $\gcd(r_k, r_{k-1})$ **takes** $k-1$ **steps and** $b> r_k> r_{k-1}$. **So since** k, $k-1 \geq 1$ **by the IH we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$.

Also, since $a \geq b$ **we must have** $q_{k+1} \geq 1$.

So $a = q_{k+1}b + r_k \geq b + r_k \geq f_{k+1}+ f_k= f_{k+2}$ **as required.** ■

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2 - 1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid's Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

so $(n-1)/2 \leq \log_2 a$ or $n \leq 1 + 2\log_2 a$
i.e., # of steps $\leq$ twice the # of bits in a.

Recursive Definition of Sets

Recursive Definition

- Basis Step: $0 \in S$
- Recursive Step: If $x \in S$, then $x + 2 \in S$
- Exclusion Rule: Every element in S follows from basis steps and a finite number of recursive steps.

even natural #s

0, 2, 4, 6, 8 . . .

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: If $x,y \in S$, then $x+y \in S$

$\{6, 12, 15, 18, 21, \ldots$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for all $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

plane in $\mathbb{R}^3$ spanned by $(1,1,0)$ $(0,1,1)$

Powers of 3:

Over $\mathbb{N}$: Basis: $3 \in S$, $1 \in S$
Recursive: If $x, y \in S$ then $x \cdot y \in S$

Basis: $1 \in S$
Recur: If $x \in S$ $3x \in S$

Over $\mathbb{R}/\mathbb{Q}$: Basis: $1 \in S$
Recursive: If $x \in S$ then $3x \in S$ and $x/3 \in S$

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: If $x,y \in S$, then $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for all $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Powers of 3:
Basis: $1 \in S$
Recursive: If $x \in S$, then $3x \in S$.

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n\right)$$

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1's before 0's

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$**, then** $0x \in S$

If $x \in S$**, then** $x1 \in S$

Function Definitions on Recursively Defined Sets

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$