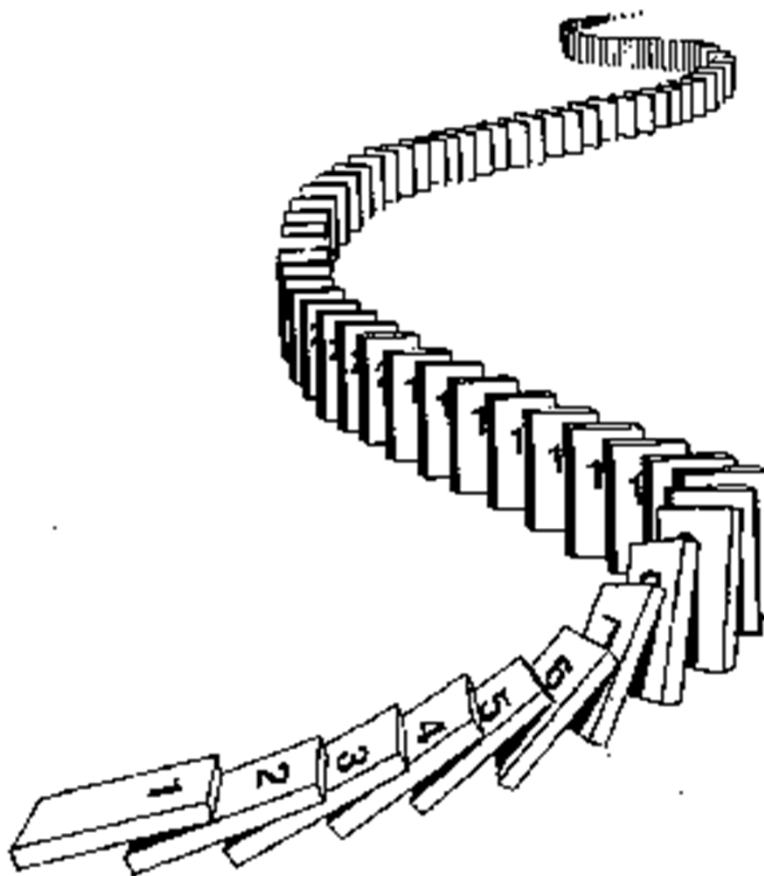


CSE 311: Foundations of Computing

Lecture 14: Induction & Strong Induction



Midterm

- **A week today (Friday, Feb 14) in class**
- **Closed book, closed notes**
 - You will get lists of inference rules & equivalences
- **Covers material up to end of ordinary induction.**
- **Practice problems & practice midterm on the website**
 - Solutions early next week
- **Solutions to HW5 in Section next Thursday**
- **I will run a review session Thursday, Feb 13, 5:00-6:30 pm in Sieg Hall 134. Please bring your questions!**

Inductive Proofs In 5 Easy Steps

1. “Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction.”
2. “Base Case:” Prove $P(0)$
3. “Inductive Hypothesis:
Assume $P(k)$ is true for some arbitrary integer $k \geq 0$ ”
4. “Inductive Step:” Prove that $P(k + 1)$ is true:
Use the goal to figure out what you need.
Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k + 1)$!!)
5. “Conclusion: $P(n)$ is true for all integers $n \geq 0$ ”

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b ?
- Define predicate $Q(k) = P(k + b)$ for all k .
 - Then $\forall n Q(n) \equiv \forall n \geq b P(n)$
- Ordinary induction for Q :
 - Prove $Q(0) \equiv P(b)$
 - Prove
$$\forall k (Q(k) \rightarrow Q(k + 1)) \equiv \forall k \geq b (P(k) \rightarrow P(k + 1))$$

Inductive Proofs In 5 Easy Steps

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Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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- 1. Let $P(n)$ be " $3^n \geq n^2 + 3$ ". We will show $P(n)$ is true for all integers $n \geq 2$ by induction.**
- 2. Base Case ($n=2$):**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let $P(n)$ be " $3^n \geq n^2 + 3$ ". We will show $P(n)$ is true for all integers $n \geq 2$ by induction.
2. Base Case ($n=2$): $3^2 = 9 \geq 7 = 4 + 3 = 2^2 + 3$ so $P(2)$ is true.
3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 2$.
4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $3^{k+1} \geq (k+1)^2 + 3$

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$$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(k^2 + 3) \text{ by the IH} \\ &= k^2 + 2k^2 + 9 \\ &\geq k^2 + 2k + 4 = (k+1)^2 + 3 \text{ since } k \geq 1. \end{aligned}$$

Therefore $P(k+1)$ is true.

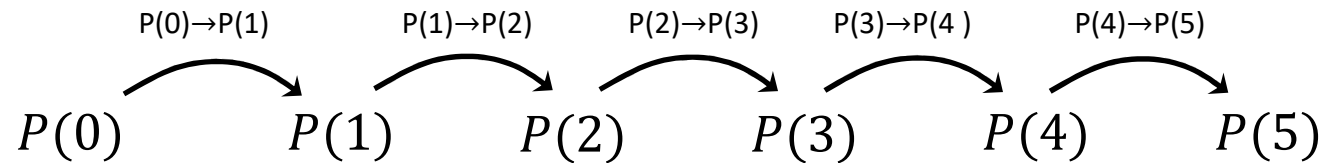
5. Thus $P(n)$ is true for all integers $n \geq 2$, by induction.

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\frac{P(0) \quad \forall k (P(k) \rightarrow P(k + 1))}{\therefore \forall n P(n)}$$

How do the givens prove $P(5)$?

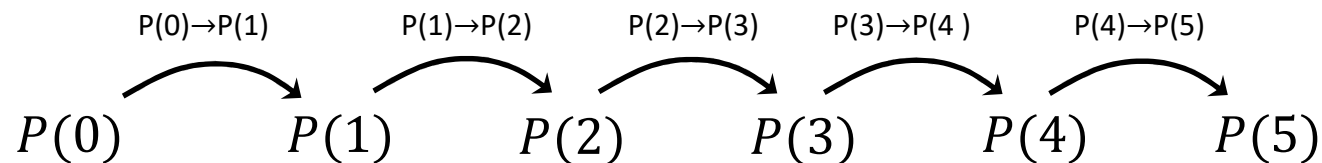


Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \hline \forall k (P(k) \rightarrow P(k + 1)) \\ \hline \therefore \forall n P(n) \end{array}$$

How do the givens prove $P(5)$?



We made it harder than we needed to ...

When we proved $P(2)$ we knew **BOTH** $P(0)$ and $P(1)$

When we proved $P(3)$ we knew $P(0)$ and $P(1)$ and $P(2)$

When we proved $P(4)$ we knew $P(0)$, $P(1)$, $P(2)$, $P(3)$

etc.

That's the essence of the idea of Strong Induction.

Strong Induction

$$P(0)$$

$$\forall k \left((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k + 1) \right)$$

$$\therefore \forall n P(n)$$

Strong Induction

$$P(0)$$

$$\forall k \left((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k + 1) \right)$$

$$\therefore \forall n P(n)$$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)$$

**Note that $Q(0) \equiv P(0)$ and $Q(k + 1) \equiv Q(k) \wedge P(k + 1)$
and $\forall n Q(n) \equiv \forall n P(n)$**

Inductive Proofs In 5 Easy Steps

1. “Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction.”
2. “Base Case:” Prove $P(b)$
3. “Inductive Hypothesis:
Assume that for some arbitrary integer $k \geq b$,
 $P(k)$ is true”
4. “Inductive Step:” Prove that $P(k + 1)$ is true:
Use the goal to figure out what you need.
Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k + 1)$!!)
5. “Conclusion: $P(n)$ is true for all integers $n \geq b$ ”

Strong Inductive Proofs In 5 Easy Steps

1. “Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by ***strong*** induction.”
2. “Base Case:” Prove $P(b)$
3. “Inductive Hypothesis:
Assume that for some arbitrary integer $k \geq b$,
 $P(j)$ is true for every integer j from b to k ”
4. “Inductive Step:” Prove that $P(k + 1)$ is true:
Use the goal to figure out what you need.
Make sure you are using I.H. (that $P(b), \dots, P(k)$ are true) and point out where you are using it.
(Don't assume $P(k + 1)$!!)
5. “Conclusion: $P(n)$ is true for all integers $n \geq b$ ”

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$591 = 3 \cdot 197$$

$$45,523 = 45,523$$

$$321,950 = 2 \cdot 5 \cdot 5 \cdot 47 \cdot 137$$

$$1,234,567,890 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 3,607 \cdot 3,803$$

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

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1. Let $P(n)$ be “ n is a product of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.
2. Base Case ($n=2$): 2 is prime, so it is a product of primes.
Therefore $P(2)$ is true.

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3. Inductive Hyp: Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer j between 2 and k
4. Inductive Step:

Goal: Show $P(k+1)$; i.e. $k+1$ is a product of primes

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Case: $k+1$ is prime: Then by definition $k+1$ is a product of primes

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Case: $k+1$ is composite: Then $k+1=ab$ for some integers a and b where $2 \leq a, b \leq k$.

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Case: $k+1$ is composite: Then $k+1=ab$ for some integers a and b where $2 \leq a, b \leq k$. By our IH, $P(a)$ and $P(b)$ are true so we have

$$a = p_1 p_2 \cdots p_r \text{ and } b = q_1 q_2 \cdots q_s$$

for some primes $p_1, p_2, \dots, p_r, q_1, q_2, \dots, q_s$.

Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ which is a product of primes.

Every integer ≥ 2 is a product of primes.

1. Let $P(n)$ be “ n is a product of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.

2. Base Case ($n=2$): 2 is prime, so it is a product of primes.
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Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ which is a product of primes.

Since $k \geq 2$, one of these cases must happen and so $P(k+1)$ is true:

5. Thus $P(n)$ is true for all integers $n \geq 2$, by strong induction.

Strong Induction is particularly useful when...

...we need to analyze methods that on input k make a recursive call for an input different from $k - 1$.

e.g.: Binary Search:

- For a problem of size $k > 1$ it makes a recursive call to a problem of size roughly $k/2$**

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0$; $F(n + 1) = F(n) + 1$ for all $n \geq 0$.
- $G(0) = 1$; $G(n + 1) = 2 \cdot G(n)$ for all $n \geq 0$.
- $0! = 1$; $(n + 1)! = (n + 1) \cdot n!$ for all $n \geq 0$.
- $H(0) = 1$; $H(n + 1) = 2^{H(n)}$ for all $n \geq 0$.

Prove $n! \leq n^n$ for all $n \geq 1$

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1. Let $P(n)$ be " $n! \leq n^n$ ". We will show that $P(n)$ is true for all integers $n \geq 1$ by induction.
2. Base Case ($n=1$): $1!=1 \cdot 0!=1 \cdot 1=1=1^1$ so $P(1)$ is true.
3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 1$.

4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(k+1)! \leq (k+1)^{k+1}$

$$\begin{aligned}(k+1)! &= (k+1) \cdot k! && \text{by definition of !} \\ &\leq (k+1) \cdot k^k && \text{by the IH and } k+1 > 0 \\ &\leq (k+1) \cdot (k+1)^k && \text{since } k \geq 0 \\ &= (k+1)^{k+1}\end{aligned}$$

Therefore $P(k+1)$ is true.

5. Thus $P(n)$ is true for all $n \geq 1$, by induction.

More Recursive Definitions

Suppose that $h: \mathbb{N} \rightarrow \mathbb{R}$.

Then we have familiar summation notation:

$$\sum_{i=0}^0 h(i) = h(0)$$

$$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^n h(i) \text{ for } n \geq 0$$

There is also product notation:

$$\prod_{i=0}^0 h(i) = h(0)$$

$$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^n h(i) \text{ for } n \geq 0$$

Fibonacci Numbers

$$f_0 = 0$$

$$f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

