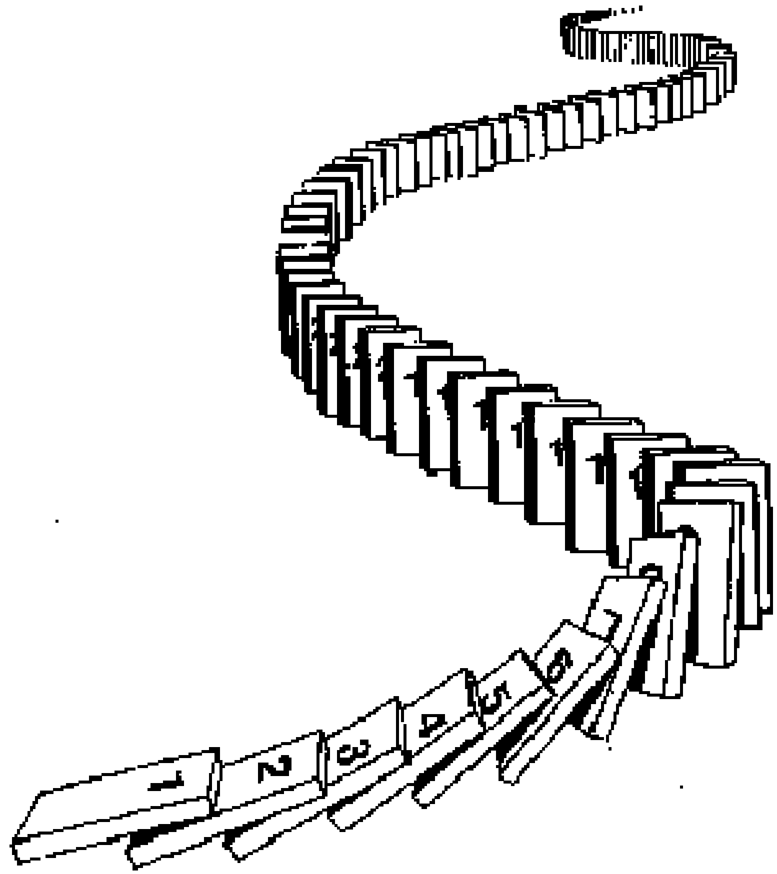
CSE 311: Foundations of Computing

Lecture 14: Induction & Strong Induction

Pick up solutions to HW 4



Midterm

- A week today (Friday, Feb 14) in class
- Closed book, closed notes
  - You will get lists of inference rules & equivalences
- Covers material up to end of ordinary induction.
- Practice problems & practice midterm on the website
  - Solutions early next week
- Solutions to HW5 in Section next Thursday

- I will run a review session Thursday, Feb 13, 5:00-6:30 pm in Sieg Hall 134. Please bring your questions!

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq 0$"

Induction: Changing the start line

- **What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?**

- **Define predicate $Q(k) = P(k+b)$ for all k.**
  - **Then** $\forall n\ Q(n) \equiv \forall n \geq b\ \ P(n)$

- **Ordinary induction for Q:**
  - **Prove** $Q(0) \equiv P(b)$
  - **Prove**

    $\forall k\ \big(Q(k) \longrightarrow Q(k+1)\big) \equiv \forall k \geq b\big(P(k) \longrightarrow P(k+1)\big)$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Prove $3^n \ge n^2 + 3$ for all $n \ge 2$

∴ !! P(n) is not a number

Proof: Let P(n) be "$3^n \ge n^2 + 3$". We prove P(n) for all integers $n \ge 2$ by induction.

2 Base Case: $n = 2$

$3^2 = 9 \ge 7 = 2^2 + 3$

∴ P(2) is true

~~$3^2 \ge 2^2 + 3$~~

~~∴ $9 \ge 7$~~ ✓ Backward reasoning

~~$2 = 3$~~

~~$0 \cdot 2 = 0 \cdot 3$~~

~~$0 = 0$~~ ✓

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2):

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$.
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3$ $= k^2+2k+4$

$3^k \geq k^2+3$ by IH (P(k))

$3^{k+1} = 3 \cdot 3^k \geq 3k^2+9$

$= k^2+2k^2+9$

$\geq k^2+2k+9$ since $k \geq 2 \geq 1$

$\geq k^2+2k+4$

$\therefore$ P(k+1) is true

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**.**
4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**.**
4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

   $3^{k+1} = 3(3^k)$

   $\geq 3(k^2+3)$ **by the IH**

   $= k^2+2k^2+9$

   $\geq k^2+2k+4 = (k+1)^2+3$ **since** $k \geq 1$.

   Therefore P(k+1) **is true.**

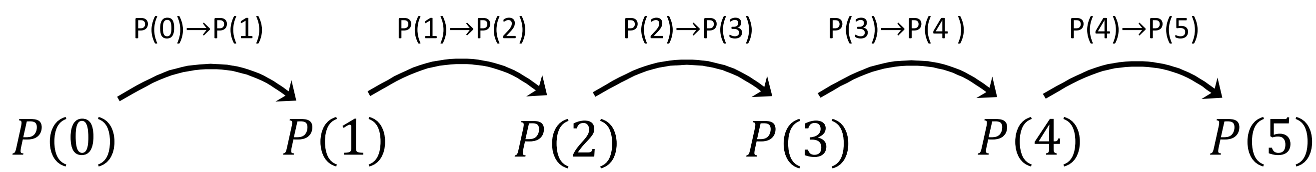
5. **Thus** P(n) **is true for all integers** $n \geq 2$, **by induction.**

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

How do the givens prove P(5)?

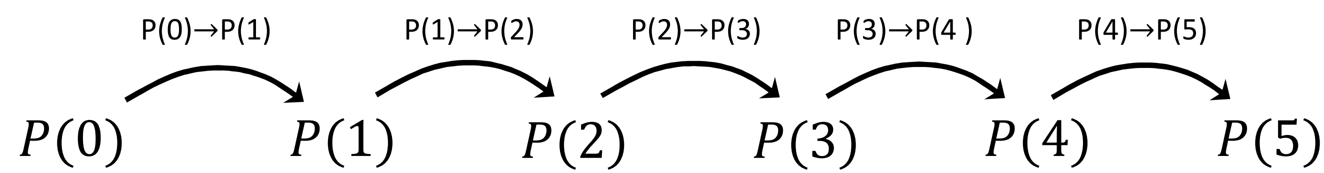


Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?



We made it harder than we needed to ...

When we proved $P(2)$ we knew BOTH $P(0)$ and $P(1)$

When we proved $P(3)$ we knew $P(0)$ and $P(1)$ and $P(2)$

When we proved $P(4)$ we knew $P(0)$, $P(1)$, $P(2)$, $P(3)$

etc.

That's the essence of the idea of Strong Induction.

Strong Induction

$$P(0)$$

$$\forall k \left(\left(P(0) \land P(1) \land P(2) \land \cdots \land P(k) \right) \to P(k+1) \right)$$

$$\therefore \forall n \, P(n)$$

$$Q(k) = P(0) \land P(1) \land \cdots \land P(k)$$

$$\equiv \exists j \left((0 \le j \le k) \to P(j) \right)$$

Strong Induction

$P(0)$

$\forall k \; \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k) \big) \to P(k+1) \Big)$

$\therefore \forall n \, P(n)$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) = P(0) \land P(1) \land P(2) \land \cdots \land P(k)$$

Note that $Q(0) \equiv P(0)$ and $Q(k+1) \equiv Q(k) \; \land P(k+1)$
and $\forall n \, Q(n) \equiv \forall n \, P(n)$

Inductive Proofs In 5 Easy Steps

1. **"Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."**
2. **"Base Case:" Prove $P(b)$**
3. **"Inductive Hypothesis:**

   Assume that for some arbitrary integer $k \geq b$,

   $P(k)$ is true"
4. **"Inductive Step:" Prove that $P(k+1)$ is true:**

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. **"Conclusion: $P(n)$ is true for all integers $n \geq b$"**

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ is true for every integer j from b to *k*"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

Proof: 1. Let $P(n)$ be "n is a product of primes"
We prove $P(n)$ for all $n \geq 2$ by strong induction

2. Base case: $(n=2)$ 2 is prime so it is a product of primes
$P(2)$ ✓

3. Inductive Hypothesis: Assume that for some integer $k \geq 2$ all integers between 2 and k are products of primes.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** k ≥ 2, P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

Case: k+1 is prime ∴ it is a product of primes

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** k ≥ 2,
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   Case: k+1 is prime: Then by definition k+1 **is a product of primes**

   Case: k+1 is composite: $k+1 = a \cdot b$ for integers a, b s.t. $1 < a < k+1$, $1 < b < k+1$
   $\therefore 2 \leq a \leq k$, $2 \leq b \leq k$
   a, b are products of primes by I.H.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** k ≥ 2**,**
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   <u>Case:</u> <u>k+1</u> **<u>is prime</u>: Then by definition** k+1 **is a product of primes**
   <u>Case:</u> <u>k+1</u> **<u>is composite:</u> Then** k+1=ab **for some integers** a **and** b
   where 2 ≤ a, b ≤ k.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$**,**
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   <u>**Case:** k+1 **is prime**</u>**: Then by definition** k+1 **is a product of primes**

   <u>**Case:** k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** P(a) **and** P(b) **are true so we have**

   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$, $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show $P(k+1)$; **i.e.** $k+1$ **is a product of primes**

   Case: $k+1$ **is prime: Then by definition** $k+1$ **is a product of primes**

   Case: $k+1$ **is composite: Then** $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** $P(a)$ **and** $P(b)$ **are true so we have**

   $a = p_1p_2 \cdots p_r$ **and** $b = q_1q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

   Since $k \geq 2$, **one of these cases must happen and so** $P(k+1)$ **is true:**
5. **Thus** $P(n)$ **is true for all integers** $n \geq 2$, **by strong induction.**

Strong Induction is particularly useful when…

…we need to analyze methods that on input k make a recursive call for an input different from $k-1$.

e.g.: Binary Search:

- For a problem of size $k > 1$ it makes a recursive call to a problem of size roughly $k/2$

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0;\ \ F(n+1) = F(n) + 1$ **for all** $n \geq\ 0$**.** ←

  $F(n) = n$

- $G(0) = 1;\ \ G(n+1) = 2 \cdot G(n)$ **for all** $n \geq\ 0$**.**

  $G(n) = 2^n$

- $0! = 1;\ \ (n+1)! = (n+1) \cdot n!$ **for all** $n \geq\ 0$**.**

- $H(0) = 1;\ \ H(n+1) = 2^{H(n)}$ **for all** $n \geq\ 0$**.**

  $H(n) = 2^{2^{2^{2^{2}}}}\Big\}\, n$

Prove $n! \leq n^n$ for all $n \geq 1$

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "$n! \leq n^n$". **We will show that** P(n) **is true for all integers** $n \geq 1$ **by induction.**
2. **Base Case** (n=1): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 1$**.**
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $(k+1)! \leq (k+1)^{k+1}$

$(k+1)! = (k+1) \cdot k!$ — **by definition of !**

$\leq (k+1) \cdot k^k$ — **by the IH and** $k+1 > 0$

$\leq (k+1) \cdot (k+1)^k$ — **since** $k \geq 0$

$= (k+1)^{k+1}$

Therefore P(k+1) **is true.**

5. **Thus** P(n) **is true for all** $n \geq 1$**, by induction.**

More Recursive Definitions

Suppose that $h\colon \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

$$\sum_{i=0}^{0} h(i) = h(0)$$

$$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^{n} h(i) \text{ for } n \geq 0$$

There is also product notation:

$$\prod_{i=0}^{0} h(i) = h(0)$$

$$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^{n} h(i) \text{ for } n \geq 0$$

$n \in \mathbb{N}$

$\sum_{i=1}^{n} h(i)$

$\sum_{i=1}^{0} h(i) = 0$

$\prod_{i=1}^{0} h(i) = 1$

Fibonacci Numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$