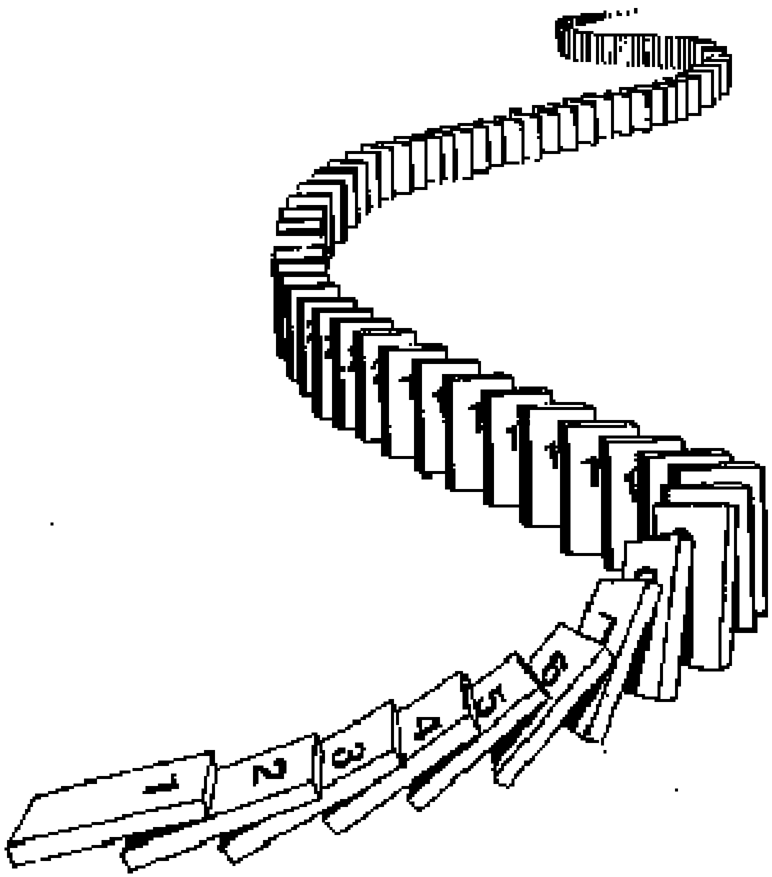
CSE 311: Foundations of Computing

Lecture 14: Induction



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x – 1) + 1; }
}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

f(3)
= f(2)+1
= f(1)+1+1
= f(0)+1+1+1
= 0+1+1+1

Prove $\forall a, b, m > 0\ \forall k \in \mathbb{N}\ (a \equiv b \pmod{m}) \to a^k \equiv b^k \pmod{m})$

Let $a, b, m > 0 \in \mathbb{Z}$ **be arbitrary. Let** $k \in \mathbb{N}$ **be arbitrary.**
Suppose that $a \equiv b \pmod{m}$**.**

We know $\left(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\right) \to a^2 \equiv b^2 \pmod{m}$ **by multiplying congruences. So, applying this repeatedly, we have:**

$$\left(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\right) \to a^2 \equiv b^2 \pmod{m}$$
$$\left(a^2 \equiv b^2 \pmod{m} \wedge a \equiv b \pmod{m}\right) \to a^3 \equiv b^3 \pmod{m}$$
$$\ldots$$
$$\left(a^{k-1} \equiv b^{k-1} \pmod{m} \wedge a \equiv b \pmod{m}\right) \to a^k \equiv b^k \pmod{m}$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$\mathbb{N} = \{0, 1, 2, \ldots\}$

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ P(n) \end{array}$$

① ②

$P(0)$ $\quad P(0) \to P(1)$ $\quad P(1) \to P(2)$ $\quad P(2) \to P(3) \ldots$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

How do the givens prove P(5)?

$P(0)$ $\quad$ $P(0) \to P(1)$ $\quad$ $P(1) \to P(2)$ $\quad$ $P(2) \to P(3)$

$P(1)$ M.P

$P(2)$

$P(3)$

. . .

Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\to P(1)} P(1) \xrightarrow{P(1)\to P(2)} P(2) \xrightarrow{P(2)\to P(3)} P(3) \xrightarrow{P(3)\to P(4)} P(4) \xrightarrow{P(4)\to P(5)} P(5)$$

First, we have P(0).
Since P(k) ⟶ P(k+1) for all k, we have P(0) ⟶ P(1).
  Since P(0) is true and P(0) ⟶ P(1), by Modus Ponens, P(1) is true.
Since P(k) ⟶ P(k+1) for all k, we have P(1) ⟶ P(2).
  Since P(1) is true and P(1) ⟶ P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

work →
⋮
P(0)
same (Let k be an arbitrary natural #
P(k) Assumption
work →
⋮
P(k+1)
same (P(k) → P(k+1) Direct Proof
∀k (P(k) → P(k+1)) Intro ∀
∀n P(n) By induction rule

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)

4. $\forall$k (P(k) $\rightarrow$ P(k+1))
5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0

3. $P(k) \rightarrow P(k+1)$
4. $\forall k\ (P(k) \rightarrow P(k+1))$ — Intro ∀: 2, 3
5. $\forall n\ P(n)$ — Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    - 3.1. Assume that P(k) is true
    - 3.2. ...
    - 3.3. Prove P(k+1) is true
3. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
4. $\forall k\ (P(k) \rightarrow P(k+1))$ — Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ — Induction: 1, 4

Translating to an English Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore\ \forall n\ \ P(n)}$$

watch

1. Prove P(0)		**Base Case**
2. Let k be an arbitrary integer ≥ 0		**Inductive Hypothesis**
3.1. Assume that P(k) is true		
3.2. ...		**Inductive Step**
3.3. Prove P(k+1) is true		
3. $P(k) \rightarrow P(k+1)$	Direct Proof Rule	**Conclusion**
4. $\forall k\,(P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3	
5. $\forall n\ P(n)$	Induction: 1, 4	

Translating To An English Proof

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**
   3.1. Assume that P(k) is true

   3.2. ... — **Inductive Step**
   3.3. Prove P(k+1) is true

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\,(P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\, P(n)$	Induction: 1, 4

Conclusion

Induction Proof Template

[...Define P(n)...]
We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.
Base Case: *[...proof of $P(0)$ here...]*
Induction Hypothesis:
  Suppose that $P(k)$ is true for some $k \in \mathbb{N}$. (arbitrary)
Induction Step:
  We want to prove that $P(k+1)$ is true.
  [...proof of $P(k+1)$ here...]
  *The proof of $P(k+1)$ **must** invoke the IH somewhere.*
So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \qquad = 1$
- $1 + 2 \qquad = 3$
- $1 + 2 + 4 \qquad = 7$
- $1 + 2 + 4 + 8 \qquad = 15$
- $1 + 2 + 4 + 8 + 16 \qquad = 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Proof (by induction)

1. Let be P(n) be "$1+2+4+\ldots+2^n = 2^{n+1}-1$"

$2^0+2^1+2^2+\ldots$

2. <u>Base case</u> $(n=0)$:

$2^0 = 1$

$2^{0+1}-1 = 2^1-1 = 2-1 = 1$

$\therefore$ P(0) is true

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

3. Inductive Hypothesis: Assume that P(k) is true for some natural # $k \geq 0$

i.e. $1+2+4+\cdots+2^k = 2^{k+1}-1$

4. Inductive Step:

$$\boxed{\begin{array}{l}\text{Goal: Show } P(k+1) \text{ is true} \\ 1+2+4+\cdots+2^{k+1} = 2^{k+2}-1\end{array}}$$

$1+2+\cdots+2^k = 2^{k+1}-1$ I.H.

$\therefore 1+2+\cdots+2^k+2^{k+1} = 2^{k+1}-1+2^{k+1}$

$= 2\cdot 2^{k+1}-1 = 2^{k+2}-1$

$1+2+\cdots+2^{k+1} = (1+2+\cdots 2^k)+2^{k+1}$

$= 2^{k+1}-1+2^{k+1}$ by IH

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + ... + 2^n = 2^{n+1} – 1". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0.
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ **by IH**

   Adding 2^{k+1} **to both sides, we get:**

   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1).

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$**.**
4. **Induction Step:**

Goal: Show $P(k+1)$**, i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$$1 + 2 + \ldots + 2^k + 2^{k+1} = (1+2+ \ldots + 2^k) + 2^{k+1}$$

$$= 2^{k+1} -1 + 2^{k+1} \quad \textbf{by the IH}$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$**, which is exactly** $P(k+1)$**.**

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**.**
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   $$1 + 2 + \ldots + 2^k + 2^{k+1} = (1+2+ \ldots + 2^k) + 2^{k+1}$$
   $$= 2^{k+1} - 1 + 2^{k+1} \text{ by the IH}$$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1)**.**
5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$, **by induction.**

Prove $1 + 2 + 3 + \dots + n = n(n+1)/2$

---

1. Let P(n) be "$1+2+3+\dots+n = n(n+1)/2$"
2. Base Case $n=0$ $LS = 0$ $=$
   $RS = 0(0+1)/2 = 0$
3. I.H.

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + ... + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0.
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** 1 + 2 + ...+ k+ (k+1) = (k+1)(k+2)/2

$$(1+2+\ldots+k)+(k+1) = \frac{k(k+1)}{2}+(k+1) \text{ by IH.}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$\therefore$ P(k+1) is true

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 1 + 2 + \ldots + k + (k+1) = (k+1)(k+2)/2}$

   $$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= (1 + 2 + \ldots + k) + (k+1) \\ &= k(k+1)/2 + (k+1) \quad \textbf{by IH} \end{aligned}$$

   Now k(k+1)/2 + (k+1) = (k+1)(k/2 + 1) = (k+1)(k+2)/2**.**

   So, we have 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2**, which is exactly** P(k+1)**.**
5. **Thus** P(n) **is true for all** n ∈ℕ**, by induction.**

Another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- $\ldots$

Prove: $3 \mid (2^{2n} - 1)$ **for all** $n \geq 0$

1. let $P(n)$ be "$3 \mid (2^{2n} - 1)$"

   we prove $P(n)$ for all integers $n \geq 0$ by induction

2. Base Case : $n = 0$ $2^{2 \cdot 0} - 1$

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "3 | (2^{2n} – 1)". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0):

$2^{2\cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$ ✓

$\therefore$ P(0) is true

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "3 | (2^{2n} – 1)". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}$-1=1-1=0=3·0 **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** 3 | ($2^{2(k+1)}$ – 1)

By I.H. $3 \mid (2^{2k}-1)$ $\therefore 2^{2k}-1 = 3m$ for some integer m.

$\therefore 2^{2k} = 3m+1$ $\therefore 2^{2(k+1)} = 2^{2k+2}$

$= 4 \cdot 2^{2k}$

$= 4\cdot(3m+1)$

$2^{2(k+1)} - 1 = 4(3m+1) - 1 = 12m+3 = 3(4m+1)$

$\therefore 3 \mid (2^{2(k+1)} - 1)$ $\therefore P(k+1)$ is true

Prove: $3 \mid (2^{2n}-1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot0}-1=1-1=0=3\cdot0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3 \mid (2^{2(k+1)} - 1)}$

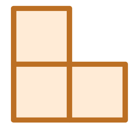
   By IH, $3 \mid (2^{2k} - 1)$ **so** $2^{2k} - 1 = 3j$ **for some integer** j

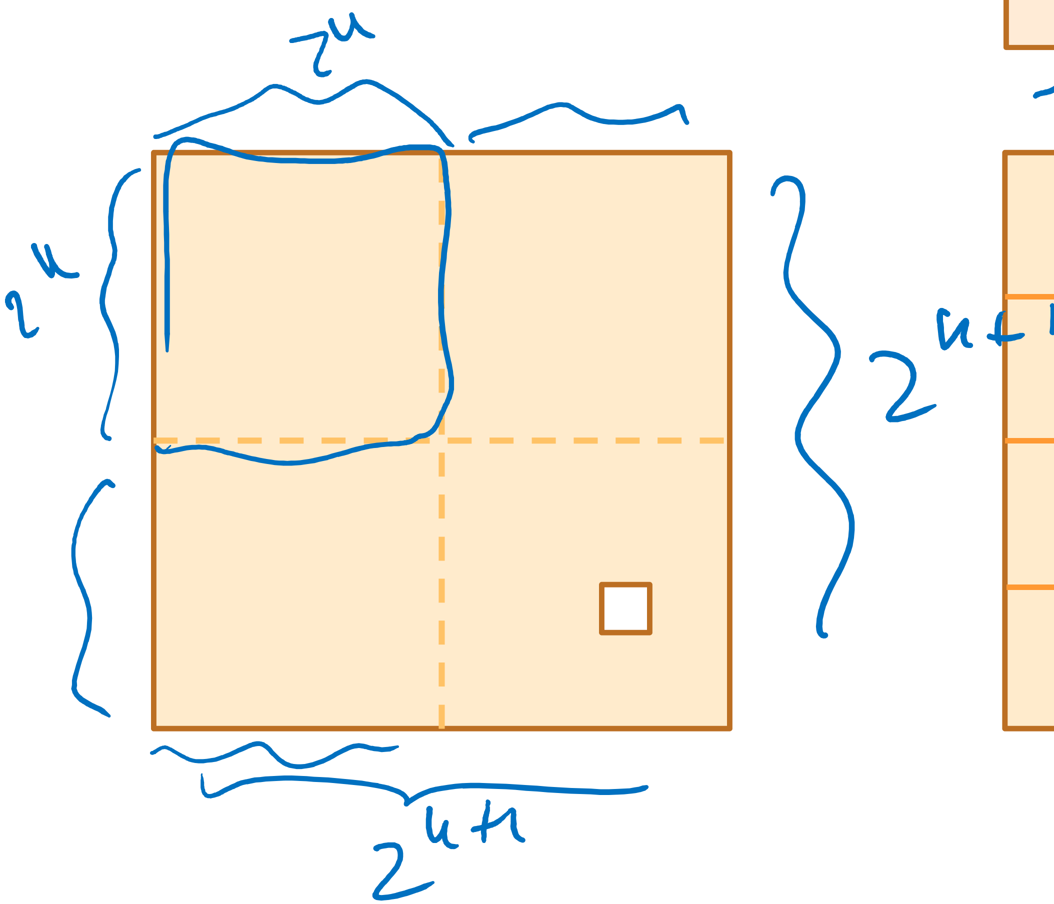
   So $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4(2^{2k}) - 1 = 4(3j+1) - 1$

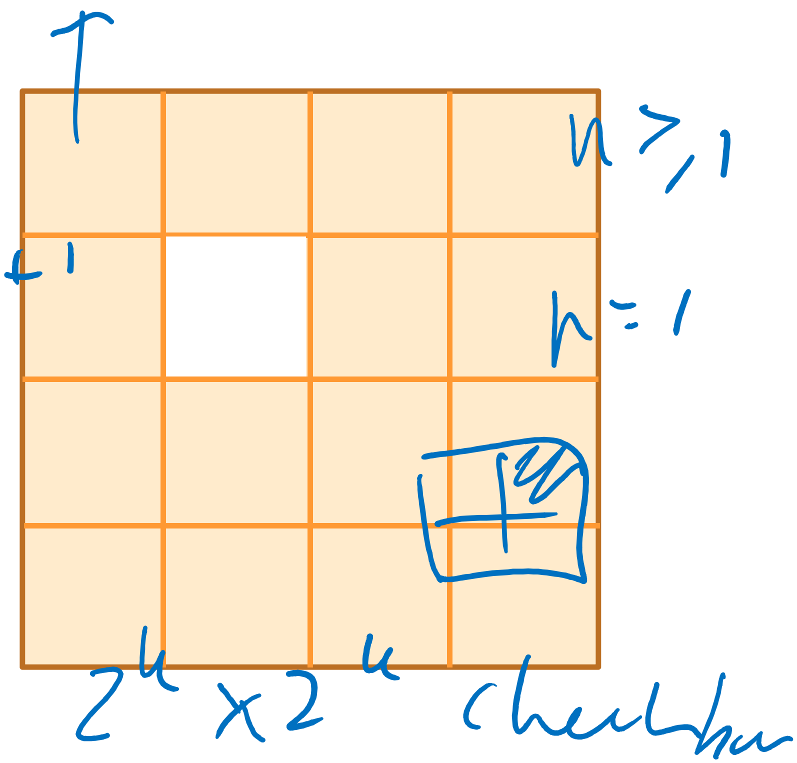
   $= 12j+3 = 3(4j+1)$

   Therefore $3 \mid (2^{2(k+1)} - 1)$ **which is exactly** P(k+1).
5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$, **by induction.**

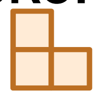
Checkerboard Tiling

- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 



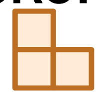


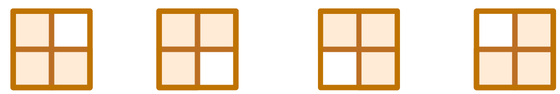
Checkerboard Tiling

1. **Let** P(n) **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove P(n) **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** n=1 

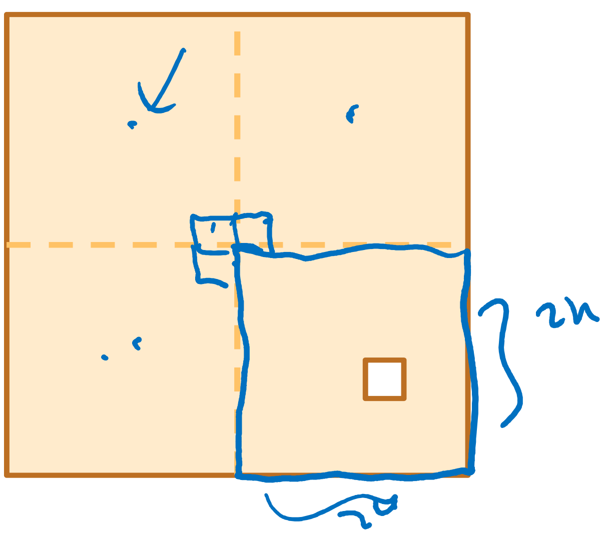
Checkerboard Tiling

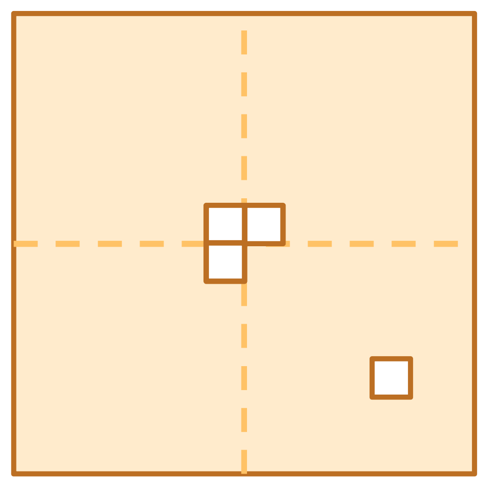
$2^{2n} - 1$

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** $n=1$



3. **Inductive Hypothesis: Assume** $P(k)$ **for some arbitrary integer** $k \geq 1$
4. **Inductive Step: Prove** $P(k+1)$





Apply IH to each quadrant then fill with extra tile.