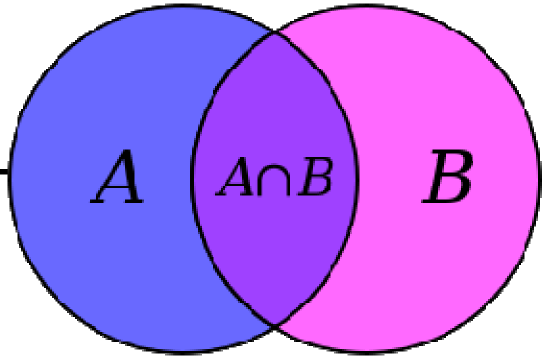
CSE 311: Foundations of Computing

Lecture 10: Set Operations & Representation, Modular Arithmetic

3902A
send email to instructor if you want to audit

Last Class: Set Theory



Sets are collections of objects called elements.

Write $a \in B$ **to say that** a **is an element of set** B**,**
and $a \notin B$ **to say that it is not.**

Some simple examples
$A = \{1\}$
$B = \{1, 3, 2\}$
$C = \{\square, 1\}$
$D = \{\{17\}, 17\}$
$E = \{1, 2, 7, \text{cat}, \text{dog}, \varnothing, \alpha\}$

Last Class: Definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\, (x \in A \rightarrow x \in B)$$

- Note: $\boxed{(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \land (\boldsymbol{B} \subseteq \boldsymbol{A})}$

Last Class: Building Sets from Predicates

S = the set of all* x for which P(x) is true

$$S = \{x : P(x)\}$$

S = the set of all x in A for which P(x) is true

$$S = \{x \in A : P(x)\}$$

*in the domain of P, usually called the "universe" U

Last Class: It's Boolean algebra again

- Definition for ∪ based on ∨

$$A \cup B = \{\, x : (x \in A) \vee (x \in B) \}$$

- Definition for ∩ based on ∧

$$A \cap B = \{\, x : (x \in A) \wedge (x \in B) \}$$

- Complement works like ¬

$$\overline{A} = \{\, x : \neg(x \in A) \}$$

Last Class: De Morgan's Laws

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

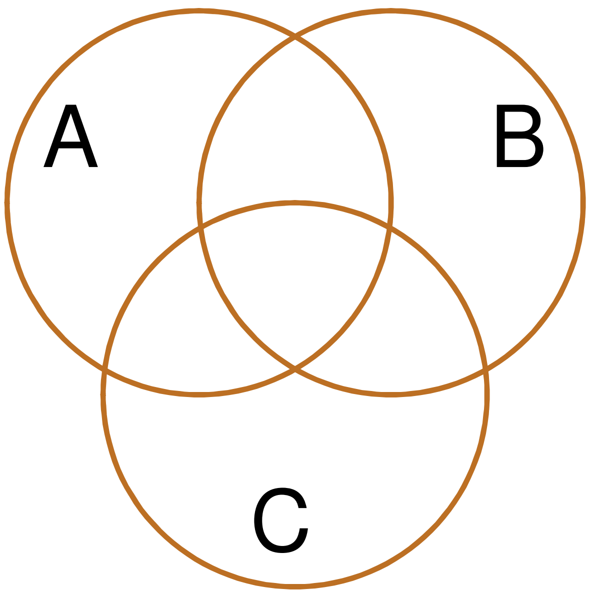
$$
\begin{aligned}
x \in \overline{A \cup B} &\equiv \neg(x \in A \cup B) && \text{Definition of } \overline{} \\
&\equiv \neg\big((x \in A) \vee (x \in B)\big) && \text{Definition of } \cup \\
&\equiv \neg(x \in A) \wedge \neg(x \in B) && \text{De Morgan} \\
&\equiv \left(x \in \overline{A}\right) \wedge \left(x \in \overline{B}\right) && \text{Definition of } \overline{} \\
&\equiv x \in \overline{A} \cap \overline{B} && \text{Definition of } \cap
\end{aligned}
$$

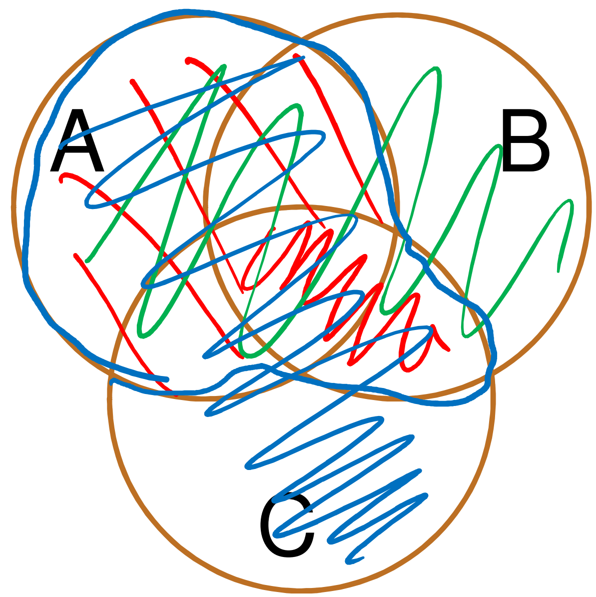
$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Similar

Last Class: Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





A Simple Set Proof

$\forall A \forall B \, (A \cap B \subseteq A)$

Prove that for any sets A and B we have $(A \cap B) \subseteq A$

Remember the definition of subset?

$A \cap B$

$$X \subseteq Y \equiv \forall x \, (x \in X \to x \in Y)$$

A

Proof: Let A and B be arbitrary sets.

Let x be arbitrary.

Suppose that $x \in A \cap B$

$\therefore$ $x \in A$ and $x \in B$ by defn

$\therefore$ $x \in A$ which is what need to prove

Since x was arbitrary. $A \cap B \subseteq A$

Since A, B were arbitrary we get the conclusion ∎

A Simple Set Proof

Prove that for any sets A and B we have $(A \cap B) \subseteq A$

Remember the definition of subset?

$$X \subseteq Y \equiv \forall x\,(x \in X \rightarrow x \in Y)$$

Proof: Let A and B be arbitrary sets and x be an arbitrary element of $A \cap B$.
Then, by definition of $A \cap B$, $x \in A$ and $x \in B$.
It follows that $x \in A$, as required. ■

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$2^3 = 8$

3

$\mathcal{P}$(Days)=? $\{\emptyset, \{M,W,F\}, \{M\}, \{W\}, \{F\}, \{M,F\}, \{M,W\}, \{W,F\}\}$

$\mathcal{P}(\emptyset)$=? $\{\emptyset\}$

0

$2^0 = 1$

$\{\emptyset\} \neq \emptyset$

Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$

$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$

Cartesian Product

$(a_1, b_1) = (a_2, b_2)$ iff $a_1 = a_2$ and $b_1 = b_2$

$$A \times B = \{\,(a,b) : a \in A, b \in B\,\}$$

$A \times B \times C = \{(a,b,c) : a \in A, b \in B, c \in C\}$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$|A|$ = # of elts of A

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

$|A \times B| = |A| \cdot |B|$

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

$$A \times \emptyset = \{(a,b) : a \in A \wedge b \in \emptyset\} = \{(a,b) : a \in A \wedge \mathrm{F}\} = \emptyset$$

$\underbrace{a \in A \wedge \mathrm{F}}_{\mathrm{F}}$

Representing Sets Using Bits

$n = 4$

$B = \{1, 4\}$

1001

- Suppose universe U is $\{1,2,\ldots,n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

$$b_1 b_2 \ldots b_n \text{ where } \quad b_i = 1 \text{ when } i \in B$$
$$b_i = 0 \text{ when } i \notin B$$

  – Called the *characteristic vector* of set B

- Given characteristic vectors for A and B
  – What is characteristic vector for $A \cup B$? $A \cap B$?

$B = \{1$

$B \cap A$

$$\begin{array}{r} 1001 \\ 1100 \\ \hline 1000 \end{array}$$

bitwise AND

$B = \{1, 4\}$

$A = \{1, 2\}$

$B \cup A =$

$$\begin{array}{r} 1001 \\ 1100 \\ \hline 1101 \end{array}$$

$\vee$ bitwise OR does $\cup$

UNIX/Linux File Permissions

- `ls -l`

```
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

- Permissions maintained as bit vectors
  - Letter means bit is 1
  - "–" means bit is 0.

chmod 755
111 101 101

Bitwise Operations

$$\begin{array}{rl} & 01101101 \\ \vee & \underline{00110111} \\ & 01111111 \end{array}$$

Java: `z=x|y`

$$\begin{array}{rl} & 00101010 \\ \wedge & \underline{00001111} \\ & 00001010 \end{array}$$

Java: `z=x&y`

$$\begin{array}{rl} & 01101101 \\ \oplus & \underline{00110111} \\ & 01011010 \end{array}$$

Java: `z=x^y`

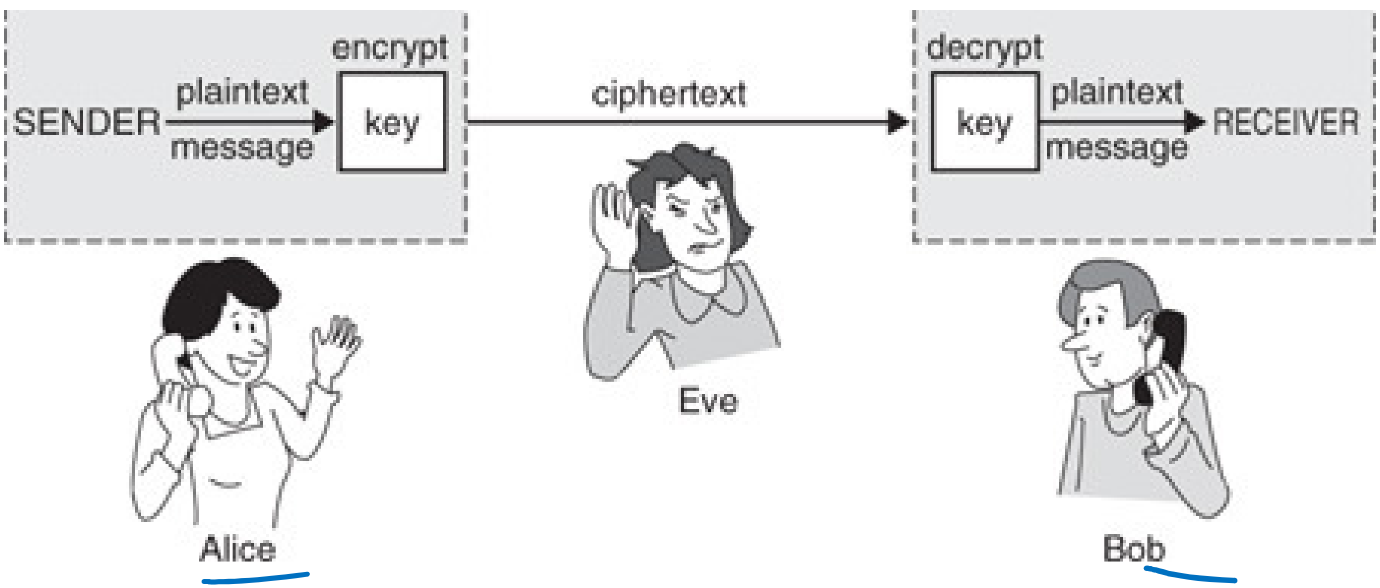
A Useful Identity

- If x and y are bits: $(x \oplus y) \oplus y = ?$ x
- What if x and y are bit-vectors?

$$\begin{array}{cc} x_1 & x_2 \\ 1 & 0 \\ 1 & 0 \\ \hline x_1 & x_2 \end{array}$$

Private Key Cryptography

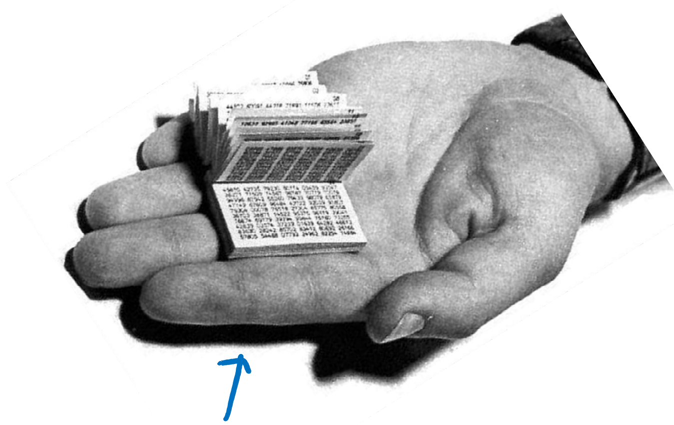
- Alice wants to communicate message secretly to Bob so that eavesdropper Eve who hears their conversation cannot tell what Alice's message is.
- Alice and Bob can get together and privately share a secret key K ahead of time.



One-Time Pad

- Alice and Bob privately share random n-bit vector K
  - Eve does not know K
- Later, Alice has n-bit message m to send to Bob
  - Alice computes $C = m \oplus K$
  - Alice sends C to Bob
  - Bob computes $m = C \oplus K$ which is $(m \oplus K) \oplus K$
- Eve cannot figure out m from C unless she can guess K

K one-time pad



Russell's Paradox

$$\boxed{S = \{ x : x \notin x \}}$$

Suppose that $S \in S$...

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose that $S \in S$. Then, by definition of S, $S \notin S$, but that's a contradiction.

Suppose that $S \notin S$. Then, by definition of the set S, $S \in S$, but that's a contradiction, too.

This is reminiscent of the truth value of the statement "This statement is false."

Number Theory (and applications to computing)

- Branch of Mathematics with direct relevance to computing

- Many significant applications
  - Cryptography
  - Hashing
  - Security

- Important tool set

Modular Arithmetic

- Arithmetic over a finite domain
- In computing, almost all computations are over a finite domain

I’m ALIVE!

```
public class Test {
    final static int SEC_IN_YEAR = 364*24*60*60*100;
    public static void main(String args[]) {
        System.out.println(
            “I will be alive for at least ” +
            SEC_IN_YEAR * 101 + “ seconds.”
        );
    }
}
```

I’m ALIVE!

```
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    public static void main(String args[]) {
        System.out.println(
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            SEC_IN_YEAR * 101 + “ seconds.”
        );
    }
}
```

```
 ----jGRASP exec: java Test
I will be alive for at least -186619904 seconds.

 ----jGRASP: operation complete.
```

Divisibility

Definition: "a divides b"

For $a \in \mathbb{Z}, b \in \mathbb{Z}$ with $a \neq 0$:

$$a \mid b \leftrightarrow \exists k \in \mathbb{Z}\ (b = ka)$$

Check Your Understanding. Which of the following are true?

$\cancel{5 \mid 1}$ $\cancel{25 \mid 5}$ $5 \mid 0$ $\cancel{3 \mid 2}$

$1 \mid 5$ $5 \mid 25$ $\cancel{0 \mid 5}$ $\cancel{2 \mid 3}$

Divisibility

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Check Your Understanding. Which of the following are true?

5 | 1	25 | 5	5 | 0	3 | 2
5 | 1 iff 1 = 5k	25 | 5 iff 5 = 25k	5 | 0 iff 0 = 5k	3 | 2 iff 2 = 3k
1 | 5	5 | 25	0 | 5	2 | 3
1 | 5 iff 5 = 1k	5 | 25 iff 25 = 5k	0 | 5 iff 5 = 0k	2 | 3 iff 3 = 2k

Division Theorem

Division Theorem

For $a \in \mathbb{Z}, d \in \mathbb{Z}$ with $d > 0$
there exist *unique* integers q, r with $0 \leq r < d$
such that $a = dq + r$.

To put it another way, if we divide d **into** a**, we get a unique quotient** $q = a$ **div** d
and non-negative remainder $r = a$ **mod** d

Note: r ≥ 0 even if a < 0.
Not quite the same as `a%d`.

Division Theorem

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To put it another way, if we divide *d* **into** *a***, we get a unique quotient** $q = a$ **div** d
and non-negative remainder $r = a$ **mod** d

$-5 = (-2)(2) + (-1)$

```
public class Test2 {
    public static void main(String args[]) {
        int a = -5;
        int d = 2;
        System.out.println(a % d);
    }
}
```

```
----jGRASP exec: java Test2
-1

----jGRASP: operation complete.
```

$-5 = (-3)(2) + 1$

Note: r ≥ 0 even if a < 0.
Not quite the same as **`a%d`**.

Arithmetic, mod 7

$$a +_7 b = (a + b) \bmod 7$$

$$a \times_7 b = (a \times b) \bmod 7$$

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Modular Arithmetic

Definition: "a is congruent to b modulo m"

For $a, b, m \in \mathbb{Z}$ with $m > 0$

$$a \equiv b \ (\mathrm{mod}\ m) \leftrightarrow m \mid (a - b)$$

Check Your Understanding. What do each of these mean? When are they true?

$2 \mid (x - 0)$

x ≡ 0 (mod 2) $\longrightarrow$ x is even $2 \mid x$

m

-1 ≡ 19 (mod 5) ✓ $5 \mid (19 - (-1))$ $5 \mid 20$

y ≡ 2 (mod 7) y is ... , -12, -5, 2, 9, 16, 23 ...

Modular Arithmetic

Definition: "a is congruent to b modulo m"

For $a, b, m \in \mathbb{Z}$ with $m > 0$

$$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$$

Check Your Understanding. What do each of these mean? When are they true?

x ≡ 0 (mod 2)

This statement is the same as saying "x is even"; so, any x that is even (including negative even numbers) will work.

-1 ≡ 19 (mod 5)

This statement is true. 19 - (-1) = 20 which is divisible by 5

y ≡ 2 (mod 7)

This statement is true for y in { ..., -12, -5, 2, 9, 16, ...}. In other words, all y of the form 2+7k for k an integer.

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
Then, $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Suppose that $a \equiv b \pmod{m}$.

$\therefore\ m \mid (a-b)$ by defn

$(a-b) = km \qquad a = km + b$

Take mod m of both sides:

$\therefore\ a \bmod m = (km + b) \bmod m = b \bmod m$

Suppose that $a \bmod m = b \bmod m$.

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
Then, $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Suppose that $a \equiv b \pmod{m}$.

Then, $m \mid (a - b)$ by definition of congruence.

So, $a - b = km$ for some integer k by definition of divides.

Therefore, $a = b + km$.

Taking both sides modulo m we get:

$$a \bmod m = (b + km) \bmod m = b \bmod m.$$

Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and

$b = ms + (b \bmod m)$ for some integers q,s.

Then, $a - b = (mq + (a \bmod m)) - (ms + (b \bmod m))$

$= m(q - s) + (a \bmod m - b \bmod m)$

$= m(q - s)$ since $a \bmod m = b \bmod m$

Therefore, $m \mid (a - b)$ and so $a \equiv b \pmod{m}$.

The $\mathrm{mod}\ m$ function vs the $\equiv \pmod{m}$ predicate

- What we have just shown
  - The $\mathrm{mod}\ m$ function takes any $a \in \mathbb{Z}$ and maps it to a remainder $a \bmod m \in \{0,1,..,m-1\}$.
  - Imagine grouping together all integers that have the same value of the $\mathrm{mod}\ m$ function

    That is, the same remainder in $\{0,1,..,m-1\}$.
  - The $\equiv \pmod{m}$ predicate compares $a, b \in \mathbb{Z}$. It is true if and only if the $\mathrm{mod}\ m$ function has the same value on a and on b.

    That is, a and b are in the same group.