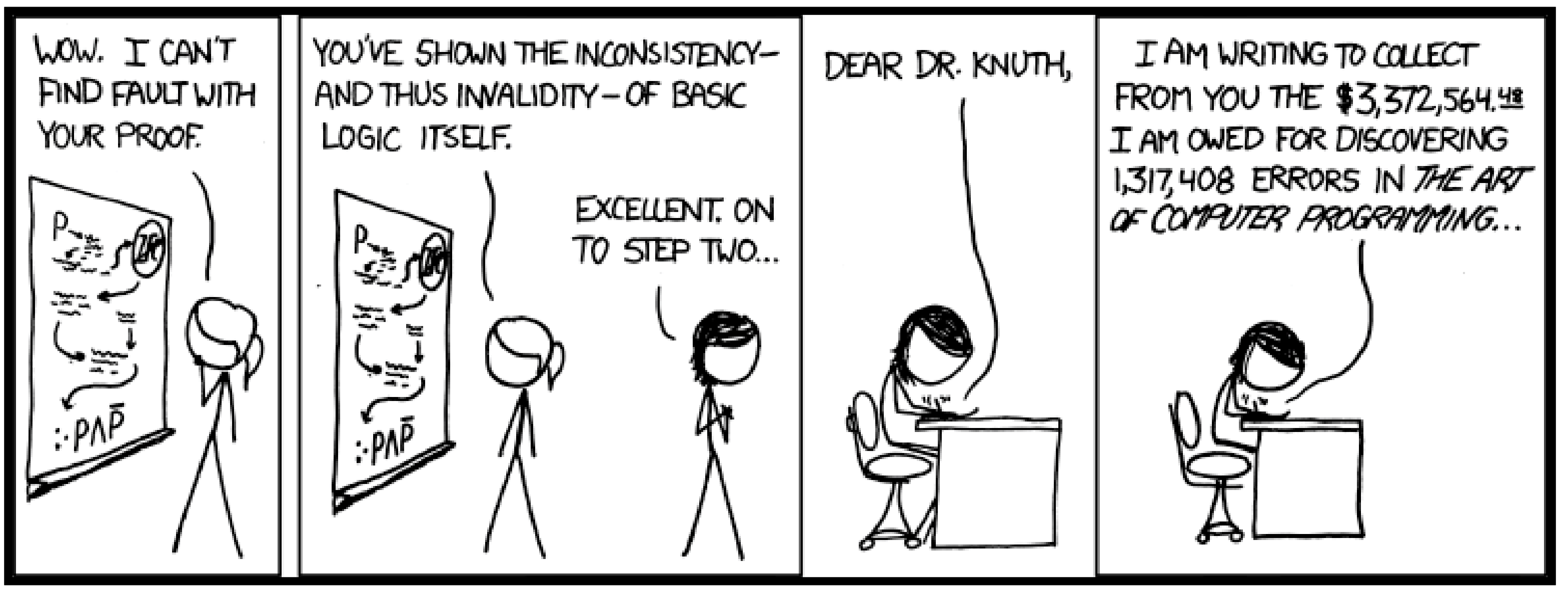
CSE 311: Foundations of Computing

Lecture 7: Logical Inference continued



Last Class: Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

Last class: An inference rule: *Modus Ponens*

- If A and $A \rightarrow B$ are both true then B must be true

- Write this rule as $$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$

- Given:
  - If it is Wednesday then you have a 311 class today.
  - It is Wednesday.

- Therefore, by Modus Ponens:
  - You have a 311 class today.

Last Class: My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

Last Class: Proofs can use equivalences too

Show that $\neg p$ follows from p ▯ q and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

Inference Rules

If A is true and B is true ....

Requirements: $\underline{\quad A \;;\; B \quad}$

Conclusions: $\therefore C \;,\; D$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \;;\; A \rightarrow B}{\therefore \quad B}$$

If I have A and $A \rightarrow B$ both true,
Then B must be true.

Axioms: Special inference rules

If I have nothing...

Requirements: ______

Conclusions: $\therefore$ C , D

Then, C must be true

Then D must be true

Example (Excluded Middle):

$$\frac{}{\therefore\ A \vee \neg A}$$

$A \vee \neg A$ must be true.

Simple Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A,\ B}$$

Intro ∧

$$\frac{A\ ;\ B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B\ ;\ \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B,\ B \vee A}$$

Modus Ponens

$$\frac{A\ ;\ A \rightarrow B}{\therefore B}$$

Direct Proof Rule

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

Proofs

Show that r follows from p, $p \quad q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$

$$\frac{A \wedge B}{\therefore\ A,\ B}$$

$$\frac{A\ ;\ B}{\therefore\ A \wedge B}$$

Proofs

Show that r follows from $p, p \to q,$ and $p \land q \to r$

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \land q$	Intro $\land$: 1, 3
5.	$p \land q \to r$	Given
6.	r	MP: 4, 5

$p \;;\; p \to q$

——— MP

$p \;;\; q$

——— Intro $\land$

$p \land q \;;\; p \land q \to r$

——— MP

r

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \to r$ given

~~2. $(p \vee q) \to r$ intro $\vee$ from 1.~~

Does not follow! e.g . $p = \mathrm{F}, q = \mathrm{T}, r = \mathrm{F}$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

First: Write down givens and goal

20. $\neg r$?

Idea: Work backwards!

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
20.	$\neg r$	MP: 2, ?

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim $\rightarrow$" which is MP.

Proofs

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove q...
  - Notice that at this point, if we prove q, we've proven $\neg r$...

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

This looks like or-elimination.

Elim ∨ $\dfrac{A \vee B \;;\; \neg A}{\therefore B}$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

18. $\neg\neg s$?
19. q $\vee$ Elim: 3, 18
20. $\neg r$ MP: 2, 19

$\neg\neg s$ doesn't show up in the givens but s does and we can use equivalences

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	?
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	$\wedge$ Elim: 1
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

No holes left! We just need to clean up a bit.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
4.	s	$\wedge$ Elim: 1
5.	$\neg\neg s$	Double Negation: 4
6.	q	$\vee$ Elim: 3, 5
7.	$\neg r$	MP: 2, 6

To Prove An Implication: $A \rightarrow B$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is a proof that "Given A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \rightarrow B$ is true

Example: Prove $p \rightarrow (p \vee q)$.

proof subroutine

Indent proof subroutine $\Rightarrow$

1.	p	Assumption
2.	$p \vee q$	Intro $\vee$: 1
3.	$p \rightarrow (p \vee q)$	Direct Proof Rule

Proofs using the direct proof rule

Show that $p \to r$ follows from q and $(p \land q) \to r$

1. q Given
2. $(p \land q) \to r$ Given

This is a proof of $p \to r$

3.1.	p	Assumption
3.2.	$p \land q$	Intro $\land$: 1, 3.1
3.3.	r	MP: 2, 3.2

If we know p is true… Then, we've shown r is true

3. $p \to r$ Direct Proof Rule

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens…

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1.	$p \wedge q$	Assumption
1.2.	p	Elim $\wedge$: 1.1
1.3.	$p \vee q$	Intro $\vee$: 1.2
1.	$(p \wedge q) \rightarrow (p \vee q)$	Direct Proof Rule

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

1.1.	$(p \rightarrow q) \wedge (q \rightarrow r)$	Assumption
1.2.	$p \rightarrow q$	∧ Elim: 1.1
1.3.	$q \rightarrow r$	∧ Elim: 1.1
1.4.1.	p	Assumption
1.4.2.	q	MP: 1.2, 1.4.1
1.4.3.	r	MP: 1.3, 1.4.2
1.4.	$p \rightarrow r$	Direct Proof Rule
1.	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Direct Proof Rule

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.

Inference Rules for Quantifiers: Easy rules

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \; P(a) \textbf{ for any } a}$$

Predicate Logic Proofs

- Can use
  - Predicate logic inference rules

    whole formulas only
  - Predicate logic equivalences (De Morgan's)

    even on subformulas
  - Propositional logic inference rules

    whole formulas only
  - Propositional logic equivalences

    even on subformulas

My First Predicate Logic Proof

Intro ∃ $\dfrac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$

Elim ∀ $\dfrac{\forall x\, P(x)}{\therefore \quad P(a) \text{ for any } a}$

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

5. $\forall x\, P(x) \rightarrow \exists x\, P(x)$?

The main connective is implication so Direct Proof Rule seems good

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$?

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃	P(c) for some c
	∴ ∃x P(x)

Elim ∀	∀x P(x)
	∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$ Intro ∃: ?

That requires P(c) for some c.

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove ∀x P(x) → ∃x P(x)

1.1. $\forall x\, P(x)$ Assumption

1.2 $P(a)$ Elim ∀: 1.1

We could have picked any name or domain expression here.

1.5. $\exists x\, P(x)$ Intro ∃: ?

That requires P(c) for some c.

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃	P(c) for some c
	∴ ∃x P(x)

Elim ∀	∀x P(x)
	∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

No holes. Just need to clean up.

1.1.	$\forall x\, P(x)$	Assumption
1.2	$P(a)$	Elim ∀: 1.1
1.5.	$\exists x\, P(x)$	Intro ∃: 1.2
1.	$\forall x\, P(x) \rightarrow \exists x\, P(x)$	Direct Proof Rule

My First Predicate Logic Proof

Intro ∃	P(c) for some c	
	∴ $\exists x\, P(x)$	

Elim ∀	$\forall x\, P(x)$	
	∴ P(a) for any a	

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1.	$\forall x\, P(x)$	Assumption
1.2	$P(a)$	Elim ∀: 1.1
1.3.	$\exists x\, P(x)$	Intro ∃: 1.2
1.	$\forall x\, P(x) \rightarrow \exists x\, P(x)$	Direct Proof Rule

Working forwards as well as backwards:
In applying "Intro ∃" rule we didn't know what expression we might be able to prove P(c) for, so we worked forwards to figure out what might work.