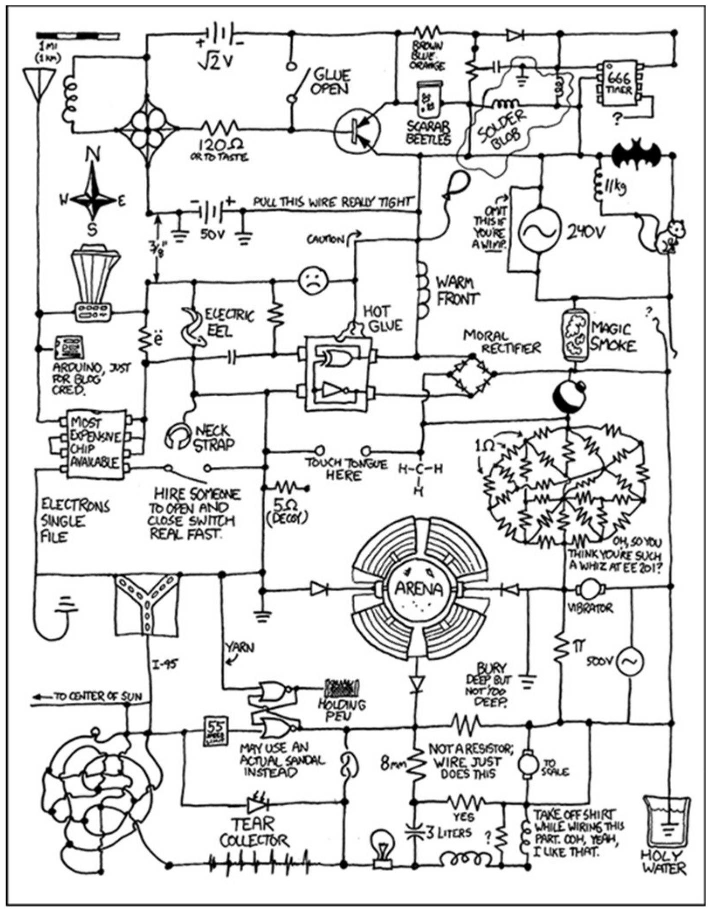
CSE 311: Foundations of Computing

Lecture 5: DNF, CNF and Predicate Logic



Administrative

- HW1 due today
  - Submit via Gradescope by 11:00 pm
  - EC1 extra credit submitted separately

- Tomorrow:
  - HW2 out
  - Quiz sections
  - 390Z/ZA sign-up still available

    Loew 113 Thursday 3:30-5:00

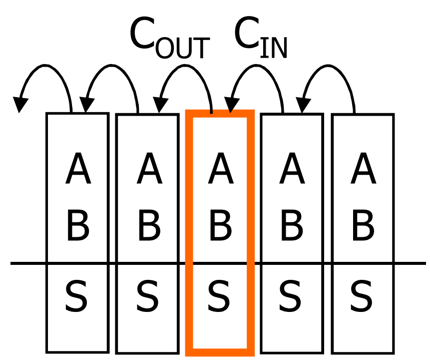
Last Class: 1-bit Binary Adder

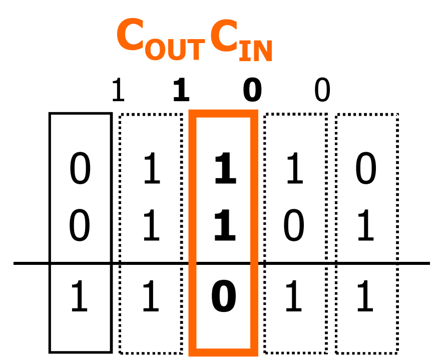
A
$\underline{+\ B}$
S
(C_{OUT})

$0 + 0 = 0$ (with $C_{OUT} = 0$)
$0 + 1 = 1$ (with $C_{OUT} = 0$)
$1 + 0 = 1$ (with $C_{OUT} = 0$)
$1 + 1 = 0$ (with $C_{OUT} = 1$)

Idea: These are chained together, with a carry-in

(C_{IN})
A
$\underline{+\ B}$
S
(C_{OUT})





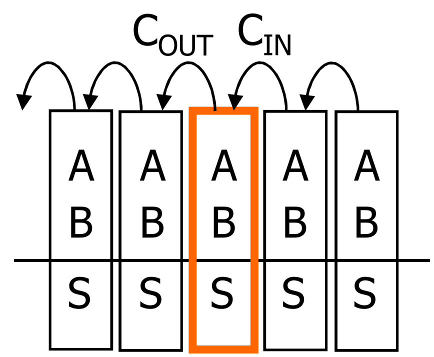
Last Class: Building Boolean Circuits

Design Process:

1. Write down a function table showing desired 0/1 inputs
2. Construct a Boolean algebra expression
   - term for each 1 in the column
   - sum (or) them to get all 1s
3. Simplify the expression using equivalences
4. Translate Boolean algebra expression to a circuit

Last Class: 1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

Last Class: Apply Theorems to Simplify Expressions

The theorems of Boolean algebra can simplify expressions

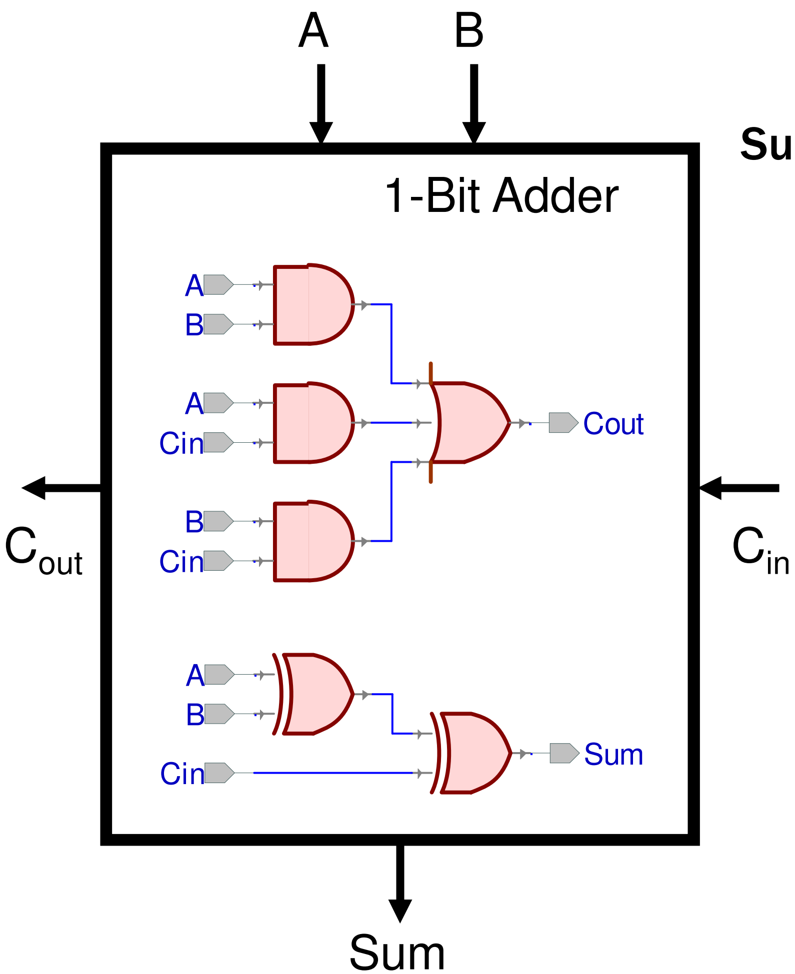
- **e.g., full adder's carry-out function**

Can simplify by combining with any one of these

$$
\begin{aligned}
\text{Cout} &= A'\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= A'\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + \boxed{A\,B\,Cin + A\,B\,Cin} \\
&= A'\,B\,Cin + A\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= (A' + A)\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= (1)\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,B'\,Cin + A\,B\,Cin' + \boxed{A\,B\,Cin + A\,B\,Cin} \\
&= B\,Cin + A\,B'\,Cin + A\,B\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,(B' + B)\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,(1)\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,Cin + A\,B\,(Cin' + Cin) \\
&= B\,Cin + A\,Cin + A\,B\,(1) \\
&= B\,Cin + A\,Cin + A\,B
\end{aligned}
$$

adding extra copies of the same term lets us reuse it for simplification

1-Bit Adder with XOR gates allowed



$$Sum = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

No Boolean algebra simplifications possible

... but Sum $\equiv$ (A $\oplus$ B) $\oplus$ C_{IN}

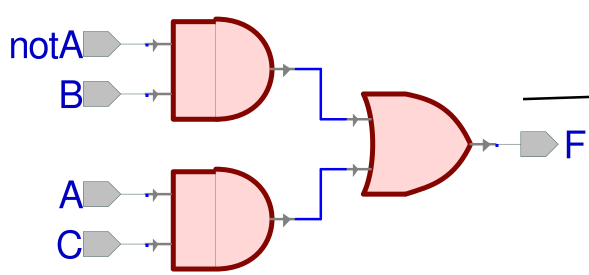
Mapping Truth Tables to Logic Gates – extra step

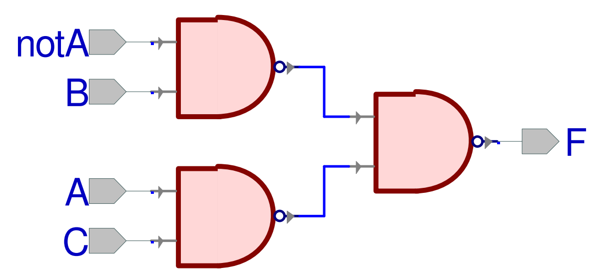
Given a truth table:

2. **Write the Boolean expression**
3. **Simplify ("minimize") the Boolean expression**
4. **Draw as gates**
5. ***Map to available gates***

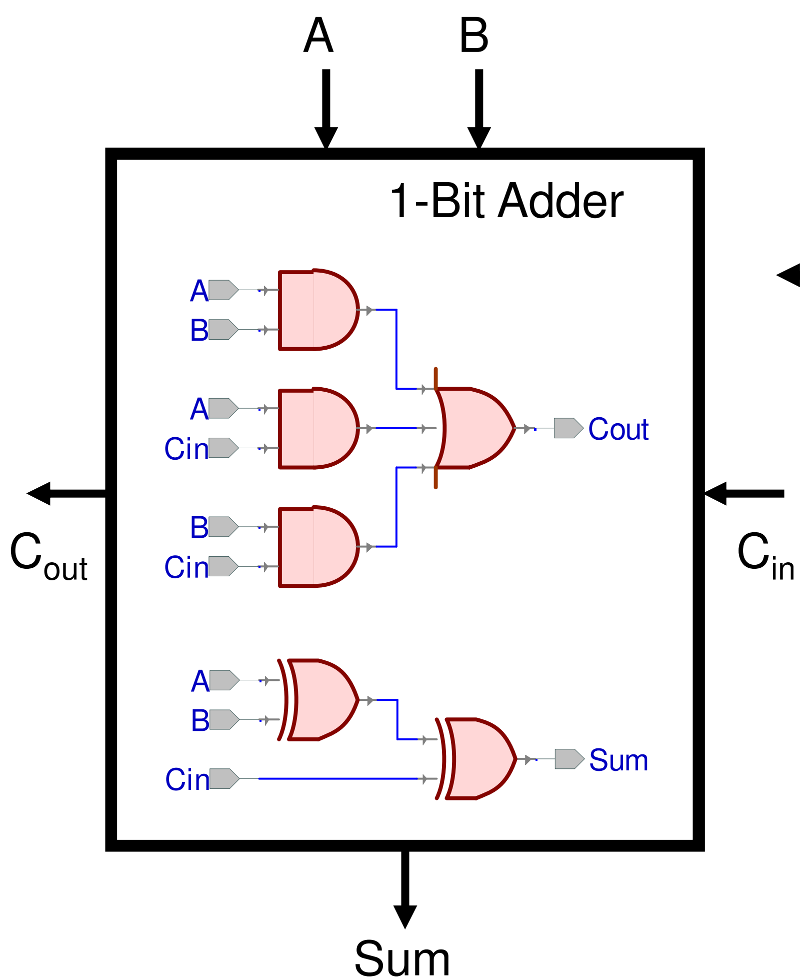
A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

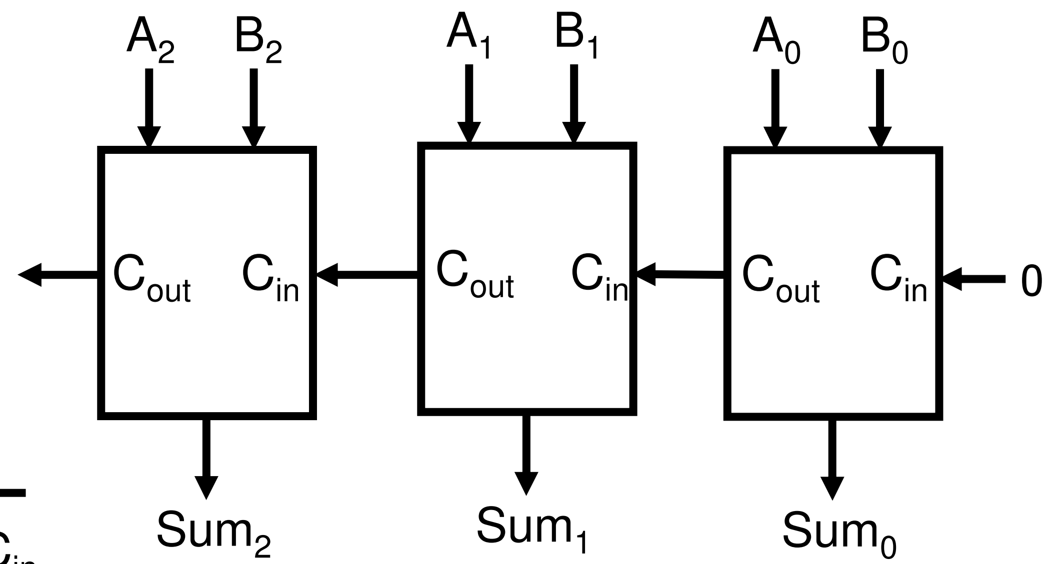
$$
\begin{aligned}
F &= A'BC'+A'BC+AB'C+ABC \\
&= A'B(C'+C)+AC(B'+B) \\
&= A'B+AC
\end{aligned}
$$





Multi-bit Ripple-Carry Adder





Canonical Forms

- Truth table is the unique signature of a Boolean Function
- The same truth table can have many gate realizations
  - We've seen this already
  - Depends on how good we are at Boolean simplification
- Canonical forms
  - Standard forms for a Boolean expression
  - We all come up with the same expression

Sum-of-Products Canonical Form

- AKA **Disjunctive Normal Form (DNF)**
- AKA **Minterm Expansion**

Don't simplify!

3 Add the (min)terms together

F= A'B'C + A'BC + AB'C + ABC' + ABC'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 Read T rows off truth table

2 Convert to Boolean Algebra

001 → A'B'C

011 → A'BC

101 → AB'C

110 → ABC'

111 → ABC

F

Sum-of-Products Canonical Form

Product term (or minterm)

- ANDed product of literals – input combination for which output is true
- each variable appears exactly once, true or inverted (but not both)

A	B	C	minterms
0	0	0	A′B′C′
0	0	1	A′B′C
0	1	0	A′BC′
0	1	1	A′BC
1	0	0	AB′C′
1	0	1	AB′C
1	1	0	ABC′
1	1	1	ABC

F in canonical form:

$$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\ &= (A'B' + A'B + AB' + AB)C + ABC' \\ &= ((A' + A)(B' + B))C + ABC' \\ &= C + ABC' \\ &= ABC' + C \\ &= AB + C \end{aligned}$$

Product-of-Sums Canonical Form

- AKA **Conjunctive Normal Form (CNF)**

4 Multiply the maxterms together

F =

1 Read F rows off truth table

2 Negate all bits

3 Convert to Boolean Algebra

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

F

Product-of-Sums Canonical Form

- AKA **Conjunctive Normal Form (CNF)**
- AKA **Maxterm Expansion**

Don't simplify!

4 Multiply the maxterms together

$F = (A + B + C)(A + B' + C)(A' + B + C)$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 Read F rows off truth table	2 Negate all bits	3 Convert to Boolean Algebra
000	111	$A + B + C$
010	101	$A + B' + C$
100	011	$A' + B + C$

F

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a minterm expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$F' = A'B'C' + A'BC' + AB'C'$

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a minterm expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Taking the complement of both sides…

$$(F')' = (A'B'C' + A'BC' + AB'C')'$$

And using DeMorgan/Comp….

$$F = (A'B'C')' \ (A'BC')' \ (AB'C')'$$

$$= (A'' + B'' + C'')(A'' + B' + C'')(A' + B'' + C'')$$

$$= (A + B + C)(A + B' + C)(A' + B + C)$$

Product-of-Sums Canonical Form

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	A+B+C
0	0	1	A+B+C′
0	1	0	A+B′+C
0	1	1	A+B′+C′
1	0	0	A′+B+C
1	0	1	A′+B+C′
1	1	0	A′+B′+C
1	1	1	A′+B′+C′

F in canonical form:

$$F(A, B, C) = (A + B + C)(A + B' + C)(A' + B + C)$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)(A + B' + C)(A' + B + C) \\ &= (A + B + C)(A + B' + C) \\ &\quad (A + B + C)(A' + B + C) \\ &= (A + C)(B + C) \end{aligned}$$

Predicate Logic

- **Propositional Logic**

  "If you take the high road and I take the low road then I'll arrive in Scotland before you."

- **Predicate Logic**

  "All positive integers x, y, and z satisfy $x^3 + y^3 \neq z^3$."

Predicate Logic

- Propositional Logic
  - Allows us to analyze complex propositions in terms of their simpler constituent parts (a.k.a. atomic propositions) joined by connectives

- Predicate Logic
  - Lets us analyze them at a deeper level by expressing how those propositions depend on the objects they are talking about

Predicate Logic

Adds two key notions to propositional logic

- Predicates
- Quantifiers

Predicates

Predicate

- **A function that returns a truth value, e.g.,**

Cat(x) ::= “x is a cat”

Prime(x) ::= “x is prime”

HasTaken(x, y) ::= “student x has taken course y”

LessThan(x, y) ::= “x < y”

Sum(x, y, z) ::= “x + y = z”

GreaterThan5(x) ::= “x > 5”

HasNChars(s, n) ::= “string s has length n”

Predicates can have varying numbers of arguments and input types.

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

(2) "x is prime", "$x = 0$", "$x < 0$", "x is a power of two"

(3) "student x has taken course y" "x is a pre-req for z"

Domain of Discourse

For ease of use, we define one “type”/“domain” that we work over. This set of objects is called the “**domain of discourse**”.

For each of the following, what might the domain be?

(1) “x is a cat”, “x barks”, “x ruined my couch”

“mammals” or “sentient beings” or “cats and dogs” or …

(2) “x is prime”, “$x = 0$”, “$x < 0$”, “x is a power of two”

“numbers” or “integers” or “integers greater than 5” or …

(3) “student x has taken course y” “x is a pre-req for z”

“students and courses” or “university entities” or …

Quantifiers

We use *quantifiers* to talk about collections of objects.

$\forall x\ P(x)$

$P(x)$ is true for every x in the domain

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x in the domain for which $P(x)$ is true

read as "there exists x, P of x"

Quantifiers

We use *quantifiers* to talk about collections of objects.

Universal Quantifier ("for all"): $\forall x\, P(x)$

$P(x)$ is true for **every** x in the domain

read as "**for all x, P of x**"

Examples:

Are these true?

- $\forall x$ Odd(x)
- $\forall x$ LessThan5(x)

Quantifiers

We use *quantifiers* to talk about collections of objects.

Universal Quantifier ("for all"): $\forall x\ P(x)$

$P(x)$ **is true for every** x **in the domain**

read as "for all x, P of x"

Examples: **Are these true? It depends on the domain. For example:**

	{1, 3, -1, -27}	Integers	Odd Integers
• $\forall$x **Odd(x)**	True	False	True
• $\forall$x **LessThan4(x)**	True	False	False

Quantifiers

We use *quantifiers* to talk about collections of objects.

Existential Quantifier ("exists"): $\exists x\, P(x)$

There is an x in the domain for which P(x) is true

read as "there exists x, P of x"

Examples: Are these true?

- $\exists x$ Odd(x)
- $\exists x$ LessThan5(x)

Quantifiers

We use *quantifiers* to talk about collections of objects.

Existential Quantifier ("exists"): $\exists x\ P(x)$

There is an x in the domain for which $P(x)$ is true

read as "there exists x, P of x"

Examples: Are these true? It depends on the domain. For example:

	{1, 3, -1, -27}	Integers	Positive Multiples of 5
• $\exists x$ Odd(x)	True	True	True
• $\exists x$ LessThan4(x)	True	True	False

Statements with Quantifiers

Just like with propositional logic, we need to define variables (this time predicates) before we do anything else. We must also now define a domain of discourse before doing anything else.

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x$ Even(x)

$\forall x$ Odd(x)

$\forall x$ (Even(x) $\vee$ Odd(x))

$\exists x$ (Even(x) $\wedge$ Odd(x))

$\forall x$ Greater(x+1, x)

$\exists x$ (Even(x) $\wedge$ Prime(x))

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x\ \mathrm{Even}(x)$	**T**	**e.g. 2, 4, 6, ...**
$\forall x\ \mathrm{Odd}(x)$	**F**	**e.g. 2, 4, 6, ...**
$\forall x\ (\mathrm{Even}(x) \vee \mathrm{Odd}(x))$	**T**	**every integer is either even or odd**
$\exists x\ (\mathrm{Even}(x) \wedge \mathrm{Odd}(x))$	**F**	**no integer is both even and odd**
$\forall x\ \mathrm{Greater}(x+1, x)$	**T**	**adding 1 makes a bigger number**
$\exists x\ (\mathrm{Even}(x) \wedge \mathrm{Prime}(x))$	**T**	**Even(2) is true and Prime(2) is true**

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

$\forall x\ \exists y\ \text{Greater}(x, y)$

$\forall x\ \exists y\ (\text{Greater}(y, x) \land \text{Prime}(y))$

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \lor \text{Odd}(x)))$

$\exists x\ \exists y\ (\text{Sum}(x, 2, y) \land \text{Prime}(x) \land \text{Prime}(y))$

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer x, there is a positive integer y, such that y > x.

$\forall x\ \exists y\ \text{Greater}(x, y)$

For every positive integer x, there is a positive integer y, such that x > y.

$\forall x\ \exists y\ (\text{Greater}(y, x) \land \text{Prime}(y))$

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \lor \text{Odd}(x)))$

For each positive integer x, if x is prime, then x = 2 or x is odd.

$\exists x\ \exists y\ (\text{Sum}(x, 2, y) \land \text{Prime}(x) \land \text{Prime}(y))$

There exist positive integers x and y such that x + 2 = y and x and y are prime.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\, \exists y\, \text{Greater}(y, x)$

There is no greatest positive integer.

$\forall x\, \exists y\, \text{Greater}(x, y)$

There is no least positive integer.

$\forall x\, \exists y\, (\text{Greater}(y, x) \land \text{Prime}(y))$

For every positive integer there is a larger number that is prime.

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \lor \text{Odd}(x)))$

Every prime number is either 2 or odd.

$\exists x\, \exists y\, (\text{Sum}(x, 2, y) \land \text{Prime}(x) \land \text{Prime}(y))$

There exist prime numbers that differ by two."

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= “x is a cat”
Red(x) ::= “x is red”
LikesTofu(x) ::= “x likes tofu”

“Red cats like tofu”

“Some red cats don’t like tofu”

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"Red cats like tofu"

$$\forall x\ ((\text{Red}(x) \wedge \text{Cat}(x)) \rightarrow \text{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\ ((\text{Red}(y) \wedge \text{Cat}(y)) \wedge \neg\text{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

When putting two predicates together like this, we use an "and".

"Red cats like tofu"

When restricting to a smaller domain in a "for all" we use implication.

When there's no leading quantification, it means "for all".

"Some red cats don't like tofu"

When restricting to a smaller domain in an "exists" we use and.

"Some" means "there exists".

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall$x PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Try your intuition! Which one "feels" right?

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall x$ PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

Domain of Discourse
{apple}

Domain of Discourse
{plum, apple}

The only choice that ensures exactly one of the statement and its negation is (b).

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

De Morgan's Laws for Quantifiers

$$\boxed{\begin{array}{c}\neg\forall x\, P(x) \equiv \exists x\, \neg P(x) \\ \neg\exists x\, P(x) \equiv \forall x\, \neg P(x)\end{array}}$$

"There is no largest integer"

$$\begin{aligned}
&\neg \exists x\, \forall y\ (x \geq y) \\
\equiv\ &\forall x\, \neg \forall y\ (x \geq y) \\
\equiv\ &\forall x\ \exists y\, \neg (x \geq y) \\
\equiv\ &\forall x\ \exists y\ (x < y)
\end{aligned}$$

"For every integer there is a larger integer"

Scope of Quantifiers

$\exists x\ (P(x) \wedge Q(x))$ **vs.** $\exists x\, P(x) \wedge \exists x\, Q(x)$

Scope of Quantifiers

$\exists x \ (P(x) \land Q(x))$ **vs.** $\exists x \, P(x) \land \exists x \, Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists y$ Greater (y, x)
$\equiv \exists z$ Greater (z, x)

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\forall x\ (\exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$$

Quantifier "Style"

$$\forall x(\exists y\ (P(x,y) \rightarrow \forall\ x\ Q(y,\ x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…