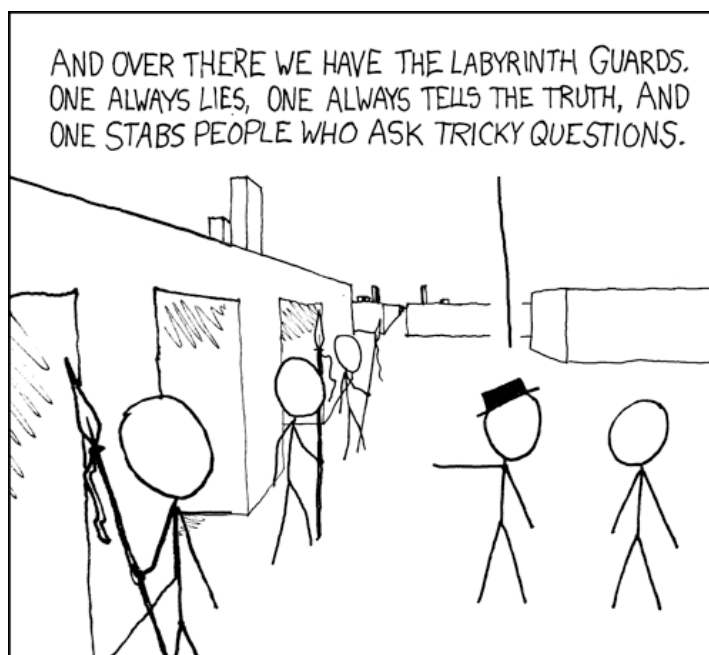


# CSE 311: Foundations of Computing

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## Lecture 4: Boolean Algebra, Circuits, Canonical Forms



# Last Class

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- More combinational logic gates
  - NAND, NOR, XOR, XNOR
- Proofs of Logical Equivalence
  - e.g.

$(p \wedge q) \rightarrow (q \vee p)$  is a tautology

$$\begin{aligned}(p \wedge q) \rightarrow (q \vee p) &\equiv \neg(p \wedge q) \vee (q \vee p) \\ &\equiv (\neg p \vee \neg q) \vee (q \vee p) \\ &\equiv \neg p \vee (\neg q \vee (q \vee p)) \\ &\equiv \neg p \vee ((\neg q \vee q) \vee p) \\ &\equiv \neg p \vee (p \vee (\neg q \vee q)) \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) \\ &\equiv (p \vee \neg p) \vee (q \vee \neg q) \\ &\equiv \mathbf{T} \vee \mathbf{T} \\ &\equiv \mathbf{T}\end{aligned}$$

#### Identity

- $p \wedge \mathbf{T} \equiv p$
- $p \vee \mathbf{F} \equiv p$

#### Domination

- $p \vee \mathbf{T} \equiv \mathbf{T}$
- $p \wedge \mathbf{F} \equiv \mathbf{F}$

#### Idempotent

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

#### Commutative

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

#### Associative

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

#### Distributive

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

#### Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

#### Negation

- $p \vee \neg p \equiv \mathbf{T}$
- $p \wedge \neg p \equiv \mathbf{F}$

Law of Implication

DeMorgan

Associative

Associative

Commutative

Associative

Commutative (twice)

Negation (twice)

Domination/Identity

# Boolean Logic: Circuits

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## Combinational Logic

- output =  $F(\text{input})$

## Sequential Logic

- $\text{output}_t = F(\text{output}_{t-1}, \text{input}_t)$ 
  - output dependent on history
  - concept of a time step (clock,  $t$ )



## Boolean Algebra: Another notation for logic consisting of...

- a set of elements  $B = \{0, 1\}$
- binary operations  $\{ + , \cdot \}$  (OR, AND)
- and a unary operation  $\{ ' \}$  (NOT)

# Boolean Algebra

---

- Usual notation used in circuit design
- Boolean algebra
  - a set of elements  $B$  containing  $\{0, 1\}$
  - binary operations  $\{ + , \cdot \}$
  - and a unary operation  $\{ ' \}$
  - such that the following axioms hold:



For any  $a, b, c$  in  $B$ :

1. closure:  $a + b$  is in  $B$
2. commutativity:  $a + b = b + a$
3. associativity:  $a + (b + c) = (a + b) + c$
4. distributivity:  $a + (b \cdot c) = (a + b) \cdot (a + c)$
5. identity:  $a + 0 = a$
6. complementarity:  $a + a' = 1$
7. null:  $a + 1 = 1$
8. idempotency:  $a + a = a$
9. involution:  $(a')' = a$

- $a \cdot b$  is in  $B$
- $a \cdot b = b \cdot a$
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- $a \cdot 1 = a$
- $a \cdot a' = 0$
- $a \cdot 0 = 0$
- $a \cdot a = a$

# A Combinational Logic Example

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## Sessions of Class:

We would like to compute the number of lectures or quiz sections remaining *at the start* of a given day of the week.

- **Inputs:** Day of the Week, Lecture/Section flag
- **Output:** Number of sessions left

Examples: Input: (Wednesday, Lecture) Output: **2**

Input: (Monday, Section) Output: **1**

# Implementation in Software

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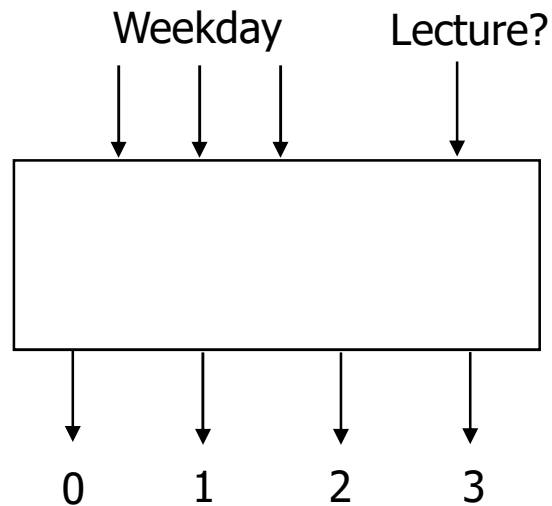
```
public int classesLeftInMorning(weekday, lecture_flag) {
    switch (weekday) {
        case SUNDAY:
        case MONDAY:
            return lecture_flag ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
            return lecture_flag ? 2 : 1;
        case THURSDAY:
            return lecture_flag ? 1 : 1;
        case FRIDAY:
            return lecture_flag ? 1 : 0;
        case SATURDAY:
            return lecture_flag ? 0 : 0;
    }
}
```

# Implementation with Combinational Logic

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## Encoding:

- How many bits for each input/output?
- Binary number for weekday
- One bit for each possible output (called “1-hot” encoding)



# Defining Our Inputs!

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## Weekday Input:

- Binary number for weekday
- Sunday = 0, Monday = 1, ...
- We care about these in binary:

Weekday	Number	Binary
Sunday	0	$(000)_2$
Monday	1	$(001)_2$
Tuesday	2	$(010)_2$
Wednesday	3	$(011)_2$
Thursday	4	$(100)_2$
Friday	5	$(101)_2$
Saturday	6	$(110)_2$



# Converting to a Truth Table!

---

```
case SUNDAY or MONDAY:
    return lecture_flag ? 3 : 1;
case TUESDAY or WEDNESDAY:
    return lecture_flag ? 2 : 1;
case THURSDAY:
    return lecture_flag ? 1 : 1;
case FRIDAY:
    return lecture_flag ? 1 : 0;
case SATURDAY:
    return lecture_flag ? 0 : 0;
```

Weekday	Lecture?	c <sub>0</sub>	c <sub>1</sub>	c <sub>2</sub>	c <sub>3</sub>
SUN	000	0			
SUN	000	1			
MON	001	0			
MON	001	1			
TUE	010	0			
TUE	010	1			
WED	011	0			
WED	011	1			
THU	100	-			
FRI	101	0			
FRI	101	1			
SAT	110	-			
-	111	-			

# Converting to a Truth Table!

---

```

case SUNDAY or MONDAY:
    return lecture_flag ? 3 : 1;
case TUESDAY or WEDNESDAY:
    return lecture_flag ? 2 : 1;
case THURSDAY:
    return lecture_flag ? 1 : 1;
case FRIDAY:
    return lecture_flag ? 1 : 0;
case SATURDAY:
    return lecture_flag ? 0 : 0;

```

Weekday	Lecture?	c <sub>0</sub>	c <sub>1</sub>	c <sub>2</sub>	c <sub>3</sub>
SUN	000	0	1	0	0
SUN	000	0	0	0	1
MON	001	0	1	0	0
MON	001	0	0	0	1
TUE	010	0	1	0	0
TUE	010	0	0	1	0
WED	011	0	1	0	0
WED	011	0	0	1	0
THU	100	0	1	0	0
FRI	101	1	0	0	0
FRI	101	0	1	0	0
SAT	110	1	0	0	0
-	111	1	0	0	0

# Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Let's begin by finding an expression for  $c_3$ . To do this, we look at the rows where  $c_3 = 1$  (true).

# Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

DAY == SUN && L == 1

DAY == MON && L == 1

# Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2d_1d_0 == 000 \ \&\& \ L == 1$

$d_2d_1d_0 == 001 \ \&\& \ L == 1$

Substituting DAY with the binary representation.

# Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2 == 0 \ \&\& \ d_1 == 0 \ \&\& \ d_0 == 0 \ \&\& \ L == 1$

$d_2 == 0 \ \&\& \ d_1 == 0 \ \&\& \ d_0 == 1 \ \&\& \ L == 1$

Splitting up the bits of the day;  
so, we can write a formula.

# Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$$d_2' \cdot d_1' \cdot d_0' \cdot L$$

$$d_2' \cdot d_1' \cdot d_0 \cdot L$$

Replacing with  
Boolean Algebra...

# Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$$d_2' \cdot d_1' \cdot d_0' \cdot L$$

$$d_2' \cdot d_1' \cdot d_0 \cdot L$$

Either situation causes  $c_3$  to be true. So, we "or" them.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$



# Truth Table to Logic (Part 2)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$		
SUN	000	0	0	1	0	0	Now, we do $c_2$ .	
SUN	000	1	0	0	0	1		
MON	001	0	0	1	0	0		
MON	001	1	0	0	0	1		
TUE	010	0	0	1	0	0		
TUE	010	1	0	0	1	0		→
WED	011	0	0	1	0	0		
WED	011	1	0	0	1	0		→
THU	100	-	0	1	0	0		
FRI	101	0	1	0	0	0		
FRI	101	1	0	1	0	0		
SAT	110	-	1	0	0	0		
-	111	-	1	0	0	0		

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

# Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	→
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	→
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	→
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	→
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	→
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	→
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

Now, we do  $c_1$ :

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

# Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0' \cdot L'$
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0 \cdot L'$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0' \cdot L'$
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0 \cdot L'$
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	???
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	$d_2 \cdot d_1' \cdot d_0 \cdot L$
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

Now, we do  $c_1$ :

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

# Truth Table to Logic (Part 3)

	$d_2 d_1 d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0' \cdot L'$
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0 \cdot L'$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0' \cdot L'$
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0 \cdot L'$
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	$d_2 \cdot d_1' \cdot d_0'$
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	$d_2 \cdot d_1' \cdot d_0 \cdot L$
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

Now, we do  $c_1$ :

No matter what L is, we always say it's 1. So, we don't need L in the expression.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

# Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0' \cdot L'$
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0 \cdot L'$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0' \cdot L'$
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0 \cdot L'$
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	$d_2 \cdot d_1' \cdot d_0'$
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	$d_2 \cdot d_1' \cdot d_0 \cdot L$
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

Now, we do  $c_1$ :

No matter what L is, we always say it's 1. So, we don't need L in the expression.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$$

# Truth Table to Logic (Part 4)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Finally, we do  $c_0$ :

$$d_2 \cdot d_1' \cdot d_0 \cdot L'$$

$$d_2 \cdot d_1 \cdot d_0'$$

$$d_2 \cdot d_1 \cdot d_0$$

# Truth Table to Logic (Part 4)

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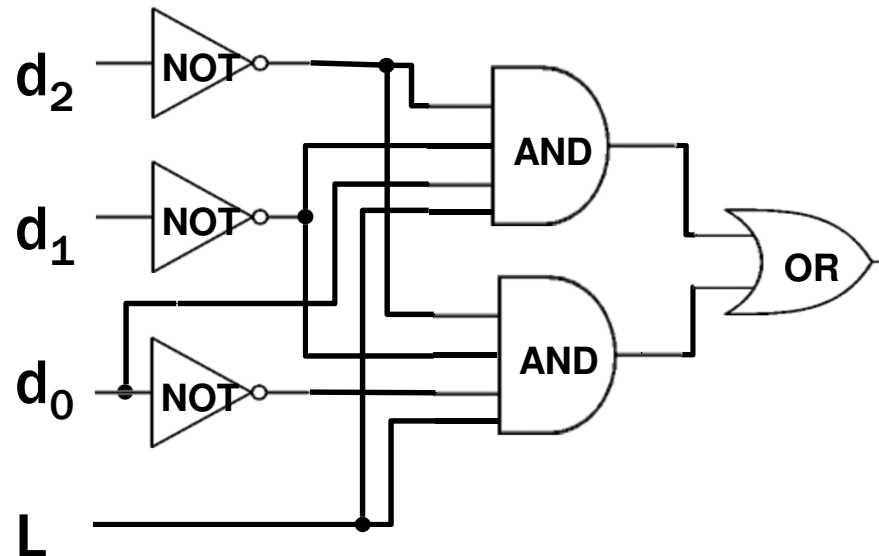
$$c_0 = d_2 \cdot d_1' \cdot d_0 \cdot L' + d_2 \cdot d_1 \cdot d_0' + d_2 \cdot d_1 \cdot d_0$$

$$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

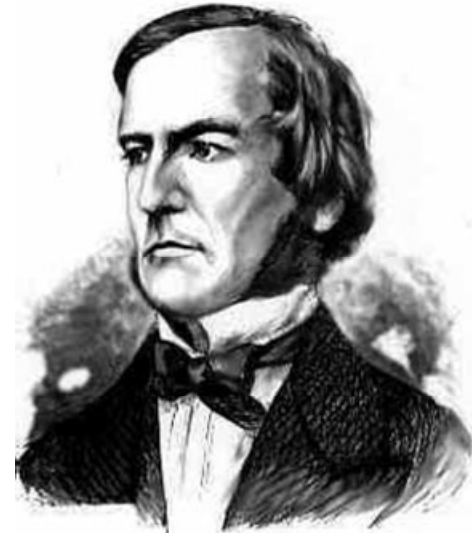
Here's  $c_3$  as a circuit:



# Boolean Algebra

---

- Usual notation used in circuit design
- Boolean algebra
  - a set of elements  $B$  containing  $\{0, 1\}$
  - binary operations  $\{ + , \cdot \}$
  - and a unary operation  $\{ ' \}$
  - such that the following axioms hold:



For any  $a, b, c$  in  $B$ :

1. closure:  $a + b$  is in  $B$
2. commutativity:  $a + b = b + a$
3. associativity:  $a + (b + c) = (a + b) + c$
4. distributivity:  $a + (b \cdot c) = (a + b) \cdot (a + c)$
5. identity:  $a + 0 = a$
6. complementarity:  $a + a' = 1$
7. null:  $a + 1 = 1$
8. idempotency:  $a + a = a$
9. involution:  $(a')' = a$

- $a \cdot b$  is in  $B$
- $a \cdot b = b \cdot a$
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- $a \cdot 1 = a$
- $a \cdot a' = 0$
- $a \cdot 0 = 0$
- $a \cdot a = a$



# Simplification using Boolean Algebra

---

## uniting:

$$10. a \cdot b + a \cdot b' = a$$

$$10D. (a + b) \cdot (a + b') = a$$

## absorption:

$$11. a + a \cdot b = a$$

$$11D. a \cdot (a + b) = a$$

$$12. (a + b') \cdot b = a \cdot b$$

$$12D. (a \cdot b') + b = a + b$$

## factoring:

$$13. (a + b) \cdot (a' + c) = \\ a \cdot c + a' \cdot b$$

$$13D. a \cdot b + a' \cdot c = \\ (a + c) \cdot (a' + b)$$

## consensus:

$$14. (a \cdot b) + (b \cdot c) + (a' \cdot c) = \\ a \cdot b + a' \cdot c$$

$$14D. (a + b) \cdot (b + c) \cdot (a' + c) = \\ (a + b) \cdot (a' + c)$$

## de Morgan's:

$$15. (a + b + \dots)' = a' \cdot b' \cdot \dots$$

$$15D. (a \cdot b \cdot \dots)' = a' + b' + \dots$$

# Proving Theorems

---

- 2. commutativity:
- 3. associativity:
- 4. distributivity:
- 5. identity:
- 6. complementarity:
- 7. null:
- 8. idempotency:
- 9. involution:

$$\begin{aligned}a + b &= b + a \\a + (b + c) &= (a + b) + c \\a + (b \cdot c) &= (a + b) \cdot (a + c) \\a + 0 &= a \\a + a' &= 1 \\a + 1 &= 1 \\a + a &= a \\(a')' &= a\end{aligned}$$

$$\begin{aligned}a \cdot b &= b \cdot a \\a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\a \cdot (b + c) &= (a \cdot b) + (a \cdot c) \\a \cdot 1 &= a \\a \cdot a' &= 0 \\a \cdot 0 &= 0 \\a \cdot a &= a\end{aligned}$$

## Using the laws of Boolean Algebra:

prove the Uniting theorem:

$$X \cdot Y + X \cdot Y' = X$$

$$X \cdot Y + X \cdot Y' =$$

prove the Absorption theorem:

$$X + X \cdot Y = X$$

$$X + X \cdot Y =$$

# Proving Theorems

---

2. commutativity:
3. associativity:
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$$\begin{aligned}a \cdot b &= b \cdot a \\ a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\ a \cdot (b + c) &= (a \cdot b) + (a \cdot c) \\ a \cdot 1 &= a \\ a \cdot a' &= 0 \\ a \cdot 0 &= 0 \\ a \cdot a &= a\end{aligned}$$

## Using the laws of Boolean Algebra:

**prove the Uniting theorem:**

$$X \cdot Y + X \cdot Y' = X$$

distributivity  
complementarity  
identity

$$\begin{aligned}X \cdot Y + X \cdot Y' &= X \cdot (Y + Y') \\ &= X \cdot 1 \\ &= X\end{aligned}$$

**prove the theorem:**

$$X + X \cdot Y = X$$

identity  
distributivity  
commutativity  
null  
identity

$$\begin{aligned}X + X \cdot Y &= X \cdot 1 + X \cdot Y \\ &= X \cdot (1 + Y) \\ &= X \cdot (Y + 1) \\ &= X \cdot 1 \\ &= X\end{aligned}$$

# Proving Theorems

---

## Using truth table:

For example, de Morgan's Law:

$(X + Y)' = X' \cdot Y'$   
NOR is equivalent to AND  
with inputs complemented

X	Y	X'	Y'	$(X + Y)'$	$X' \cdot Y'$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

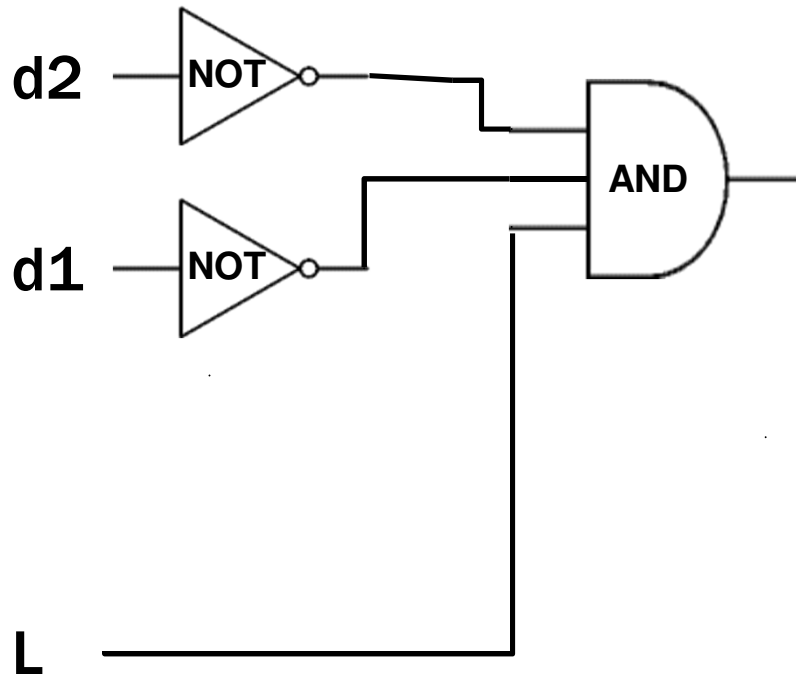
$(X \cdot Y)' = X' + Y'$   
NAND is equivalent to OR  
with inputs complemented

X	Y	X'	Y'	$(X \cdot Y)'$	$X' + Y'$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

# Simplifying using Boolean Algebra

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$$\begin{aligned}c3 &= d2' \cdot d1' \cdot d0' \cdot L + d2' \cdot d1' \cdot d0 \cdot L \\ &= d2' \cdot d1' \cdot (d0' + d0) \cdot L \\ &= d2' \cdot d1' \cdot 1 \cdot L \\ &= d2' \cdot d1' \cdot L\end{aligned}$$



# 1-bit Binary Adder

---

A	$0 + 0 = 0$ (with $C_{OUT} = 0$ )
<u>+ B</u>	$0 + 1 = 1$ (with $C_{OUT} = 0$ )
S	$1 + 0 = 1$ (with $C_{OUT} = 0$ )
( $C_{OUT}$ )	$1 + 1 = 0$ (with $C_{OUT} = 1$ )

# 1-bit Binary Adder

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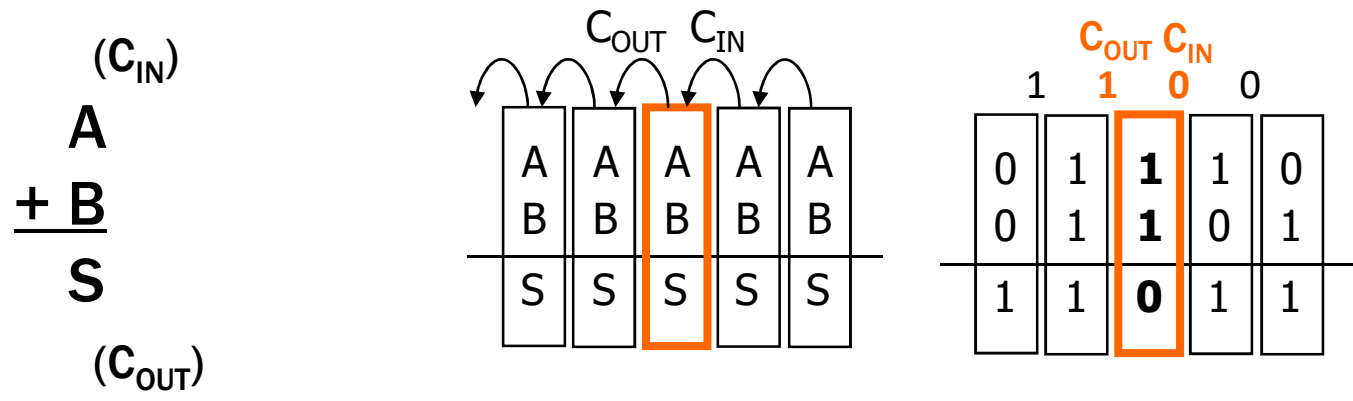
A	$0 + 0 = 0$ (with $C_{OUT} = 0$ )
<u>+ B</u>	$0 + 1 = 1$ (with $C_{OUT} = 0$ )
S	$1 + 0 = 1$ (with $C_{OUT} = 0$ )
( $C_{OUT}$ )	$1 + 1 = 0$ (with $C_{OUT} = 1$ )

**Idea: To chain these together, let's add a carry-in**

# 1-bit Binary Adder

A	$0 + 0 = 0$ (with $C_{OUT} = 0$ )
<u>+ B</u>	$0 + 1 = 1$ (with $C_{OUT} = 0$ )
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( $C_{OUT}$ )	$1 + 1 = 0$ (with $C_{OUT} = 1$ )

Idea: To chain these together, let's add a carry-in

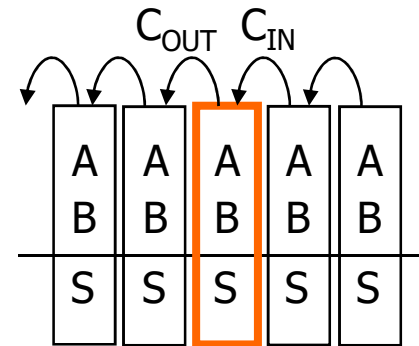




# 1-bit Binary Adder

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- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out

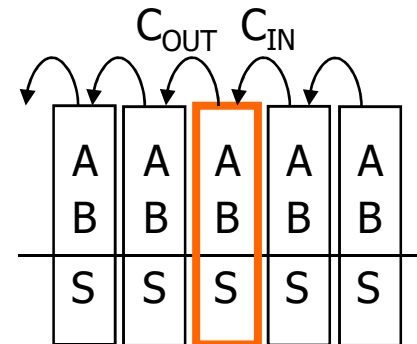


A	B	C <sub>IN</sub>	C <sub>OUT</sub>	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



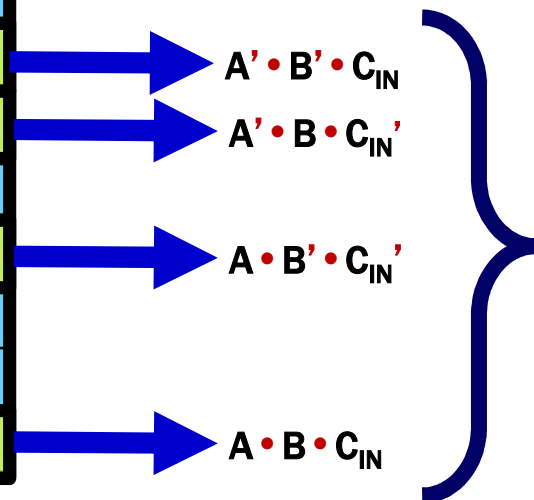
# 1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C <sub>IN</sub>	C <sub>OUT</sub>	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

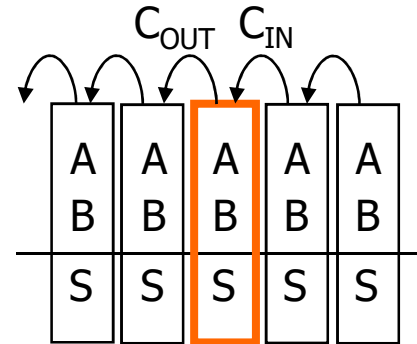
Derive an expression for S



$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

# 1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C <sub>IN</sub>	C <sub>OUT</sub>	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for C<sub>OUT</sub>

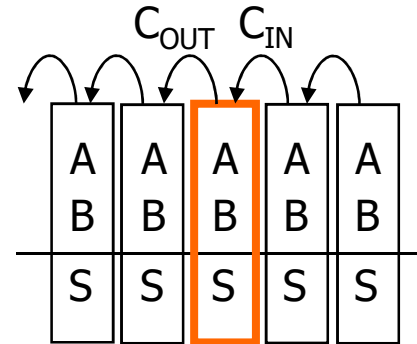
$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

The diagram shows four blue arrows pointing from the rows of the truth table to the terms in the equation above. The first arrow points from the row (0, 1, 1) to the term  $A' \cdot B \cdot C_{IN}$ . The second arrow points from the row (1, 0, 1) to the term  $A \cdot B' \cdot C_{IN}$ . The third arrow points from the row (1, 1, 0) to the term  $A \cdot B \cdot C_{IN}'$ . The fourth arrow points from the row (1, 1, 1) to the term  $A \cdot B \cdot C_{IN}$ . A large blue curly bracket groups these four terms together, pointing to the equation.

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

# 1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	$C_{IN}$	$C_{OUT}$	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

# Apply Theorems to Simplify Expressions

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The theorems of Boolean algebra can simplify expressions

– e.g., full adder's carry-out function

$$\begin{aligned} \text{Cout} &= A' B \text{Cin} + A B' \text{Cin} + A B \text{Cin}' + A B \text{Cin} \\ &= A' B \text{Cin} + A B' \text{Cin} + A B \text{Cin}' + \boxed{A B \text{Cin} + A B \text{Cin}} \\ &= A' B \text{Cin} + A B \text{Cin} + A B' \text{Cin} + A B \text{Cin}' + A B \text{Cin} \\ &= (A' + A) B \text{Cin} + A B' \text{Cin} + A B \text{Cin}' + A B \text{Cin} \\ &= (1) B \text{Cin} + A B' \text{Cin} + A B \text{Cin}' + A B \text{Cin} \\ &= B \text{Cin} + A B' \text{Cin} + A B \text{Cin}' + \boxed{A B \text{Cin} + A B \text{Cin}} \\ &= B \text{Cin} + A B' \text{Cin} + A B \text{Cin} + A B \text{Cin}' + A B \text{Cin} \\ &= B \text{Cin} + A (B' + B) \text{Cin} + A B \text{Cin}' + A B \text{Cin} \\ &= B \text{Cin} + A (1) \text{Cin} + A B \text{Cin}' + A B \text{Cin} \\ &= B \text{Cin} + A \text{Cin} + A B (\text{Cin}' + \text{Cin}) \\ &= B \text{Cin} + A \text{Cin} + A B (1) \\ &= B \text{Cin} + A \text{Cin} + A B \end{aligned}$$

adding extra terms  
creates new factoring  
opportunities