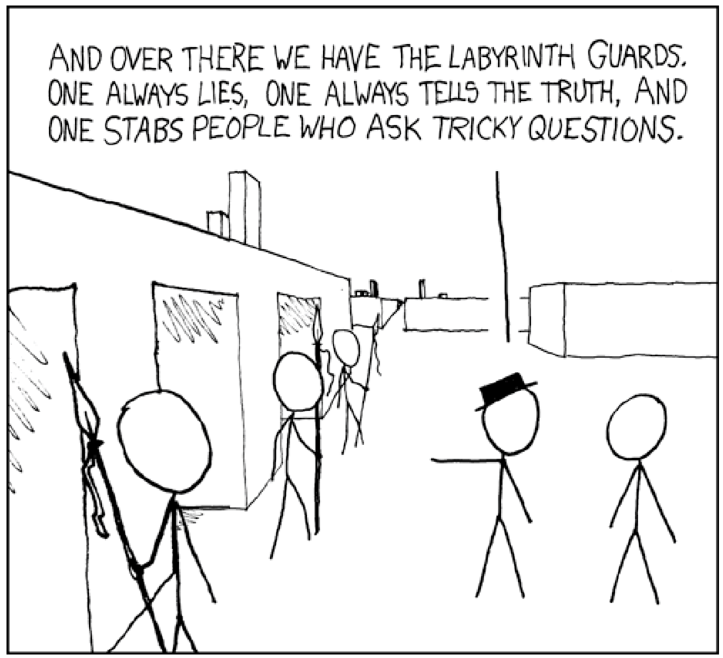
CSE 311: Foundations of Computing

Lecture 4: Boolean Algebra, Circuits, Canonical Forms



Last Class

- **More combinational logic gates**
  - NAND, NOR, XOR, XNOR
- **Proofs of Logical Equivalence**
  - e.g.

Associative
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Absorption
- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation
- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

$(p \wedge q) \rightarrow (q \vee p)$ **is a tautology**

$(p \wedge q) \rightarrow (q \vee p) \equiv \neg(p \wedge q) \vee (q \vee p)$	Law of Implication
$\equiv (\neg p \vee \neg q) \vee (q \vee p)$	DeMorgan
$\equiv \neg p \vee (\neg q \vee (q \vee p))$	Associative
$\equiv \neg p \vee ((\neg q \vee q) \vee p)$	Associative
$\equiv \neg p \vee (p \vee (\neg q \vee q))$	Commutative
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	Associative
$\equiv (p \vee \neg p) \vee (q \vee \neg q)$	Commutative (twice)
$\equiv \mathbf{T} \vee \mathbf{T}$	Negation (twice)
$\equiv \mathbf{T}$	Domination/Identity

Identity
- $p \wedge \mathrm{T} \equiv p$
- $p \vee \mathrm{F} \equiv p$

Domination
- $p \vee \mathrm{T} \equiv \mathrm{T}$
- $p \wedge \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

Commutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Boolean Logic: Circuits

4up

Combinational Logic

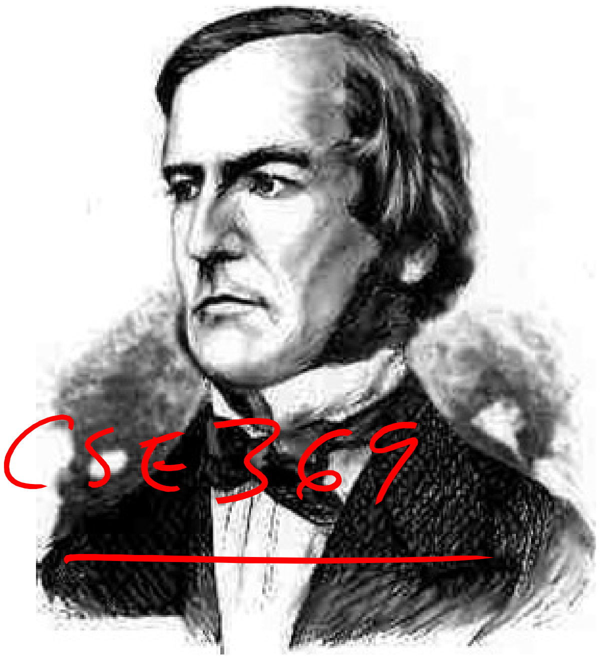
– output = F(input)

Sequential Logic

– $output_t = F(output_{t-1}, input_t)$

- output dependent on history
- concept of a time step (clock, t)

CSE 369



CSE 369

Boolean Algebra: Another notation for logic consisting of…

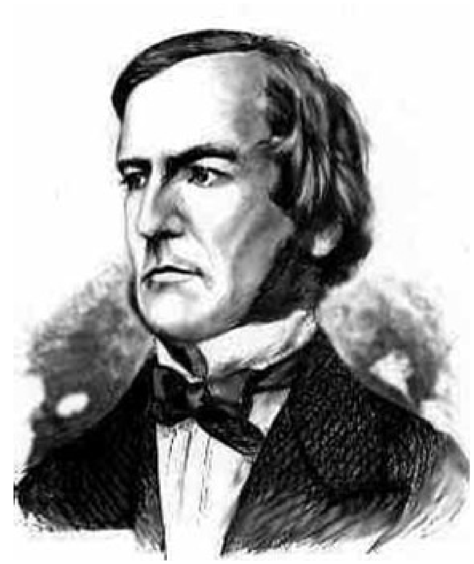
– a set of elements B = {0, 1}

– binary operations { + , • } (OR, AND)

– and a unary operation { ’ } (NOT)

Boolean Algebra

- Usual notation used in circuit design
- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:



For any a, b, c in B:

1. closure:	a + b is in B	a • b is in B
2. commutativity:	a + b = b + a	a • b = b • a
3. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
4. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
5. identity:	a + 0 = a	a • 1 = a
6. complementarity:	a + a’ = 1	a • a’ = 0
7. null:	a + 1 = 1	a • 0 = 0
8. idempotency:	a + a = a	a • a = a
9. involution:	(a’)’ = a	

A Combinational Logic Example

Sessions of Class:

We would like to compute the number of lectures or quiz sections remaining *at the start* of a given day of the week.

- **Inputs:** Day of the Week, Lecture/Section flag
- **Output:** Number of sessions left

Examples: Input: (Wednesday, Lecture) Output: 2

Input: (Monday, Section) Output: 1

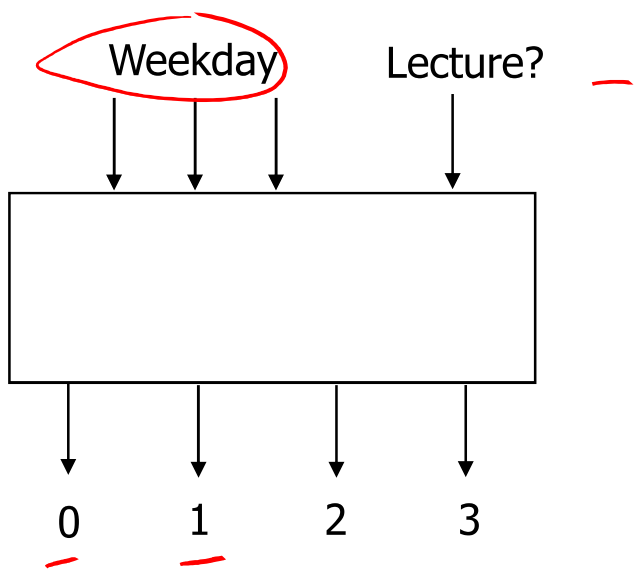
Implementation in Software

```
public int classesLeftInMorning(weekday, lecture_flag) {
    switch (weekday) {
        case SUNDAY:
        case MONDAY:
            return lecture_flag ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
            return lecture_flag ? 2 : 1;
        case THURSDAY:
            return lecture_flag ? 1 : 1;
        case FRIDAY:
            return lecture_flag ? 1 : 0;
        case SATURDAY:
            return lecture_flag ? 0 : 0;
  }
}
```

Implementation with Combinational Logic

Encoding:

- How many bits for each input/output?
- Binary number for weekday
- One bit for each possible output (called "1-hot" encoding)



1000 0
0100 1
0010 2
0001 3

Defining Our Inputs!

Weekday Input:

- Binary number for weekday
- Sunday = 0, Monday = 1, ...
- We care about these in binary:

Weekday	Number	Binary
Sunday	0	$(000)_2$
Monday	1	$(001)_2$
Tuesday	2	$(010)_2$
Wednesday	3	$(011)_2$
Thursday	4	$(100)_2$
Friday	5	$(101)_2$
Saturday	6	$(110)_2$

Converting to a Truth Table!

```
case SUNDAY or MONDAY:
    return lecture_flag ? 3 : 1;
case TUESDAY or WEDNESDAY:
    return lecture_flag ? 2 : 1;
case THURSDAY:
    return lecture_flag ? 1 : 1;
case FRIDAY:
    return lecture_flag ? 1 : 0;
case SATURDAY:
    return lecture_flag ? 0 : 0;
```

Weekday		Lecture?	c_0	c_1	c_2	c_3
SUN	000	0				
SUN	000	1				
MON	001	0				
MON	001	1				
TUE	010	0				
TUE	010	1	0	0	1	0
WED	011	0				
WED	011	1	0	0	1	0
THU	100	-				
FRI	101	0				
FRI	101	1				
SAT	110	-				
-	111	-				

Converting to a Truth Table!

```
case SUNDAY or MONDAY:
    return lecture_flag ? 3 : 1;
case TUESDAY or WEDNESDAY:
    return lecture_flag ? 2 : 1;
case THURSDAY:
    return lecture_flag ? 1 : 1;
case FRIDAY:
    return lecture_flag ? 1 : 0;
case SATURDAY:
    return lecture_flag ? 0 : 0;
```

Weekday		Lecture?	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Let's begin by finding an expression for c_3. To do this, we look at the rows where $c_3 = 1$ (true).

c_3 = "DAY=SUN" • L

+ "DAY=MON" • L

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

DAY == SUN && L == 1

DAY == MON && L == 1

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2d_1d_0$ == 000 && L == 1

$d_2d_1d_0$ == 001 && L == 1

Substituting DAY with the binary representation.

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2' \cdot d_1' \cdot d_0' \cdot L$

$d_2 == 0$ && $d_1 == 0$ && $d_0 == 0$ && $L == 1$

$d_2 == 0$ && $d_1 == 0$ && $d_0 == 1$ && $L == 1$

Splitting up the bits of the day; so, we can write a formula.

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2' \bullet d_1' \bullet d_0' \bullet L$

$d_2' \bullet d_1' \bullet d_0 \bullet L$

Replacing with Boolean Algebra...

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2' \cdot d_1' \cdot d_0' \cdot L$

$d_2' \cdot d_1' \cdot d_0 \cdot L$

Either situation causes c_3 to be true. So, we "or" them.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Truth Table to Logic (Part 2)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Now, we do c_2.

DAY=TUE && L= 1

$d_2' \cdot d_1 \cdot d_0' \cdot L$

$+\ d_2' \cdot d_1 \cdot d_0 \cdot L$

$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Now, we do c_1:

$d_2' \cdot d_1' \cdot d_0' \cdot L'$

$+$

$c_3 = d_2' \bullet d_1' \bullet d_0' \bullet L + d_2' \bullet d_1' \bullet d_0 \bullet L$

$c_2 = d_2' \bullet d_1 \bullet d_0' \bullet L + d_2' \bullet d_1 \bullet d_0 \bullet L$

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Now, we do c_1:

$d_2' \cdot d_1' \cdot d_0' \cdot L'$

$d_2' \cdot d_1' \cdot d_0 \cdot L'$

$d_2' \cdot d_1 \cdot d_0' \cdot L'$

$d_2' \cdot d_1 \cdot d_0 \cdot L'$

??? $d_2 \cdot d_1' \cdot d_0'$

$d_2 \cdot d_1' \cdot d_0 \cdot L$

$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$

Truth Table to Logic (Part 3)

Now, we do c_1:

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3	
SUN	000	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0' \cdot L'$
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0 \cdot L'$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0' \cdot L'$
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0 \cdot L'$
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	$d_2 \cdot d_1' \cdot d_0'$
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	$d_2 \cdot d_1' \cdot d_0 \cdot L$
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

No matter what L is, we always say it's 1. So, we don't need L in the expression.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$
$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3	
SUN	000	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0' \cdot L'$
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	$d_2' \cdot d_1' \cdot d_0 \cdot L'$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0' \cdot L'$
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	$d_2' \cdot d_1 \cdot d_0 \cdot L'$
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	$d_2 \cdot d_1' \cdot d_0'$
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	$d_2 \cdot d_1' \cdot d_0 \cdot L$
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

Now, we do c_1:

No matter what L is, we always say it's 1. So, we don't need L in the expression.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$$

Truth Table to Logic (Part 4)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Finally, we do c_0:

$d_2 \cdot d_1' \cdot d_0 \cdot L'$

$d_2 \cdot d_1 \cdot d_0'$

$d_2 \cdot d_1 \cdot d_0$

Truth Table to Logic (Part 4)

$c_0 = d_2 \cdot d_1' \cdot d_0 \cdot L' + d_2 \cdot d_1 \cdot d_0' + d_2 \cdot d_1 \cdot d_0$

$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$

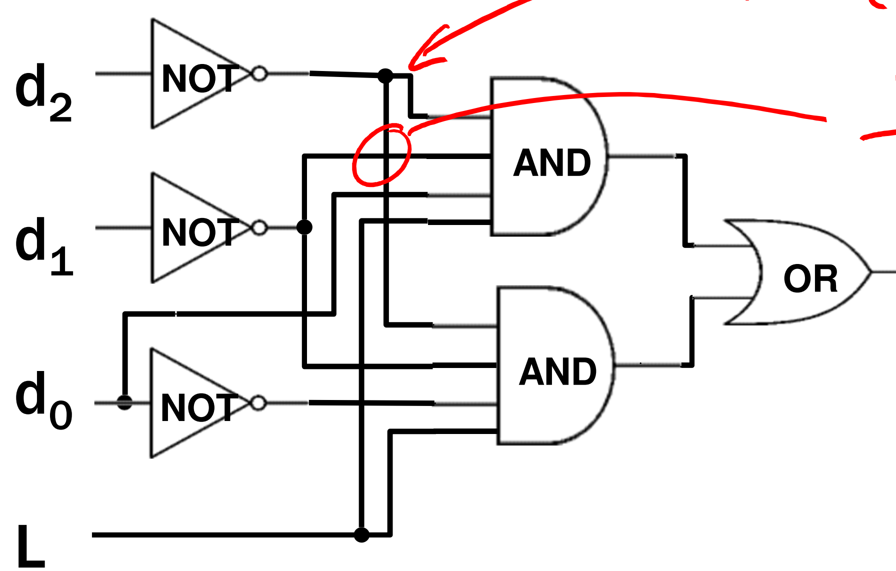
$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$

$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

Here's c_3 as a circuit:

connected

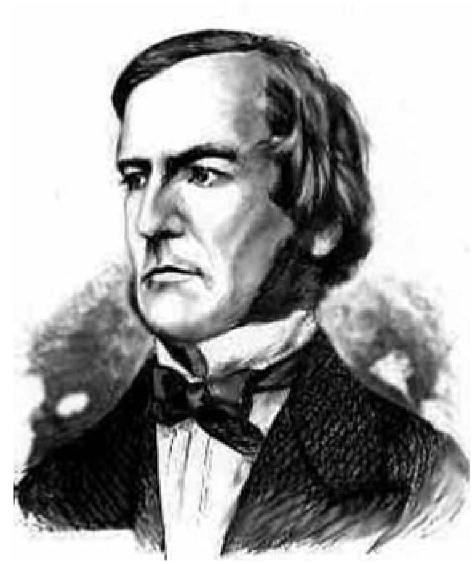
not connected



Boolean Algebra

- Usual notation used in circuit design

- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:



For any a, b, c in B:

1. closure:	a + b is in B	a • b is in B
2. commutativity:	a + b = b + a	a • b = b • a
3. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
4. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
5. identity:	a + 0 = a	a • 1 = a
6. complementarity:	a + a’ = 1	a • a’ = 0
7. null:	a + 1 = 1	a • 0 = 0
8. idempotency:	a + a = a	a • a = a
9. involution:	(a’)’ = a	

Simplification using Boolean Algebra

uniting:

10. $a \cdot b + a \cdot b' = a$

10D. $(a + b) \cdot (a + b') = a$

absorption:

11. $a + a \cdot b = a$

11D. $a \cdot (a + b) = a$

12. $(a + b') \cdot b = a \cdot b$

12D. $(a \cdot b') + b = a + b$

factoring:

13. $(a + b) \cdot (a' + c) = a \cdot c + a' \cdot b$

13D. $a \cdot b + a' \cdot c = (a + c) \cdot (a' + b)$

consensus:

14. $(a \cdot b) + (b \cdot c) + (a' \cdot c) = a \cdot b + a' \cdot c$

14D. $(a + b) \cdot (b + c) \cdot (a' + c) = (a + b) \cdot (a' + c)$

de Morgan's:

15. $(a + b + \ldots)' = a' \cdot b' \cdot \ldots$

15D. $(a \cdot b \cdot \ldots)' = a' + b' + \ldots$

Proving Theorems

2. commutativity:	$a + b = b + a$	$a \bullet b = b \bullet a$
3. associativity:	$a + (b + c) = (a + b) + c$	$a \bullet (b \bullet c) = (a \bullet b) \bullet c$
4. distributivity:	$a + (b \bullet c) = (a + b) \bullet (a + c)$	$a \bullet (b + c) = (a \bullet b) + (a \bullet c)$
5. identity:	$a + 0 = a$	$a \bullet 1 = a$
6. complementarity:	$a + a' = 1$	$a \bullet a' = 0$
7. null:	$a + 1 = 1$	$a \bullet 0 = 0$
8. idempotency:	$a + a = a$	$a \bullet a = a$
9. involution:	$(a')' = a$	

Using the laws of Boolean Algebra:

prove the Uniting theorem: $X \bullet Y + X \bullet Y' = X$

$X \bullet Y + X \bullet Y' = X \cdot (Y + Y')$ dist

$= X \cdot 1$ comple

$= X$ ident

prove the Absorption theorem: $X + X \bullet Y = X$

$X + X \bullet Y = X \cdot 1 + X \cdot Y$ ident

$= X \cdot (1 + Y)$ dist

$= X \cdot (Y + 1)$ comm

$= X \cdot 1$ null

$= X$ ide

Proving Theorems

2. commutativity:	$a + b = b + a$	$a \bullet b = b \bullet a$
3. associativity:	$a + (b + c) = (a + b) + c$	$a \bullet (b \bullet c) = (a \bullet b) \bullet c$
4. distributivity:	$a + (b \bullet c) = (a + b) \bullet (a + c)$	$a \bullet (b + c) = (a \bullet b) + (a \bullet c)$
5. identity:	$a + 0 = a$	$a \bullet 1 = a$
6. complementarity:	$a + a' = 1$	$a \bullet a' = 0$
7. null:	$a + 1 = 1$	$a \bullet 0 = 0$
8. idempotency:	$a + a = a$	$a \bullet a = a$
9. involution:	$(a')' = a$	

Using the laws of Boolean Algebra:

prove the Uniting theorem: $X \bullet Y + X \bullet Y' = X$

distributivity	$X \bullet Y + X \bullet Y' = X \bullet (Y + Y')$
complementarity	$= X \bullet 1$
identity	$= X$

prove the theorem: $X + X \bullet Y = X$

identity	$X + X \bullet Y = X \bullet 1 + X \bullet Y$
distributivity	$= X \bullet (1 + Y)$
commutativity	$= X \bullet (Y + 1)$
null	$= X \bullet 1$
identity	$= X$

Proving Theorems

Using truth table:

For example, de Morgan's Law:

$(X + Y)' = X' \bullet Y'$

NOR is equivalent to AND with inputs complemented

X	Y	X′	Y′	(X + Y)′	X′ • Y′
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

$(X \bullet Y)' = X' + Y'$

NAND is equivalent to OR with inputs complemented

X	Y	X′	Y′	(X • Y)′	X′ + Y′
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

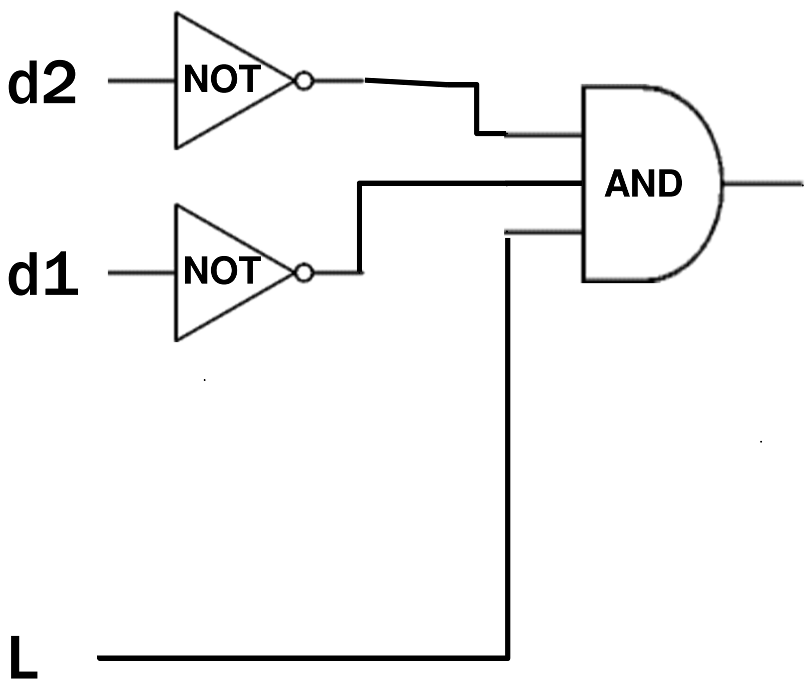
Simplifying using Boolean Algebra

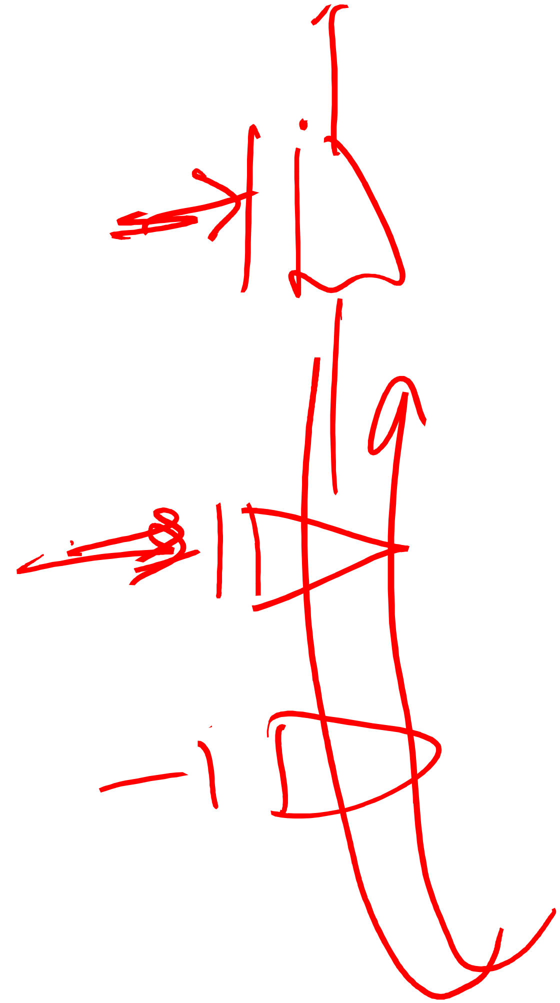
c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

= d2'•d1'•(d0' + d0)•L

= d2'•d1'•1•L

= d2'•d1'•L





1-bit Binary Adder

A	1	$0 + 0 = \underline{0}$ (with $C_{OUT} = 0$)
+ B	1	$0 + 1 = \underline{1}$ (with $C_{OUT} = 0$)
S	10	$1 + 0 = \underline{1}$ (with $C_{OUT} = 0$)
(C_{OUT})		$1 + 1 = \underline{0}$ (with $C_{OUT} = 1$)

1

1-bit Binary Adder

A	$0 + 0 = 0$ (with $C_{OUT} = 0$)
+ B	$0 + 1 = 1$ (with $C_{OUT} = 0$)
S	$1 + 0 = 1$ (with $C_{OUT} = 0$)
(C_{OUT})	$1 + 1 = 0$ (with $C_{OUT} = 1$)

Idea: To chain these together, let's add a carry-in

1-bit Binary Adder

A	$0 + 0 = 0$ (with $C_{OUT} = 0$)
$\underline{+\ B}$	$0 + 1 = 1$ (with $C_{OUT} = 0$)
S	$1 + 0 = 1$ (with $C_{OUT} = 0$)
(C_{OUT})	$1 + 1 = 0$ (with $C_{OUT} = 1$)

Idea: To chain these together, let's add a carry-in

(C_{IN})

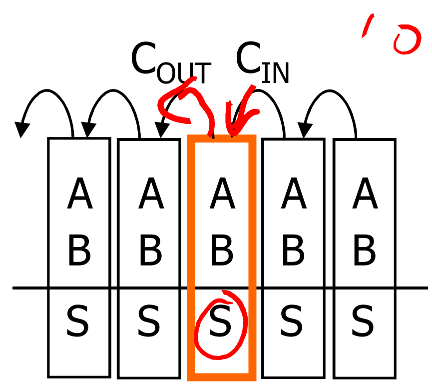
A

$\underline{+\ B}$

S

(C_{OUT})

C_{out}	S
0	0
0	1
1	0
1	1



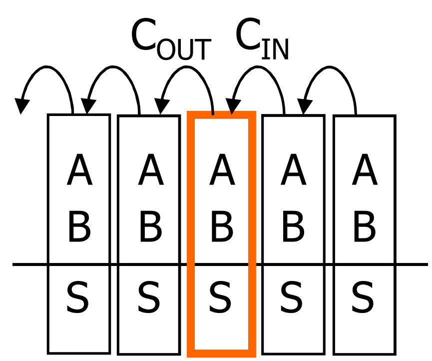
1-bit Binary Adder

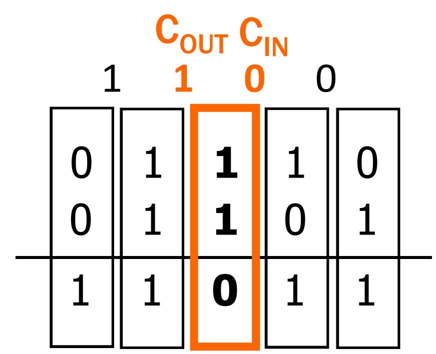
$$\begin{array}{r} A \\ \underline{+\ B} \\ S \\ (C_{OUT}) \end{array}$$

$0 + 0 = 0$ (with $C_{OUT} = 0$)
$0 + 1 = 1$ (with $C_{OUT} = 0$)
$1 + 0 = 1$ (with $C_{OUT} = 0$)
$1 + 1 = 0$ (with $C_{OUT} = 1$)

Idea: To chain these together, let's add a carry-in

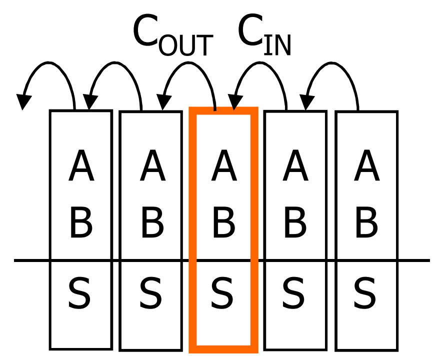
$$\begin{array}{r} (C_{IN}) \\ A \\ \underline{+\ B} \\ S \\ (C_{OUT}) \end{array}$$





1-bit Binary Adder

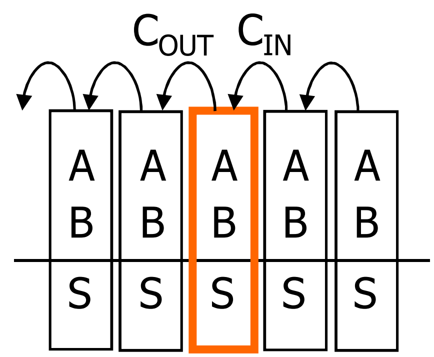
- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



Inputs outputs

A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for S

$A' \cdot B' \cdot C_{IN}$

$A' \cdot B \cdot C_{IN}'$

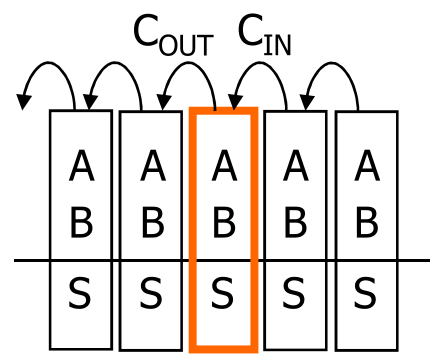
$A \cdot B' \cdot C_{IN}'$

$A \cdot B \cdot C_{IN}$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for C_{OUT}

$A' \cdot B \cdot C_{IN}$

$A \cdot B' \cdot C_{IN}$

$A \cdot B \cdot C_{IN}'$

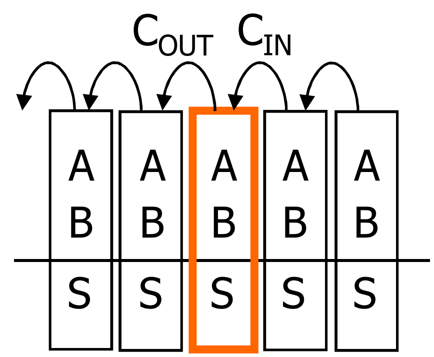
$A \cdot B \cdot C_{IN}$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

Apply Theorems to Simplify Expressions

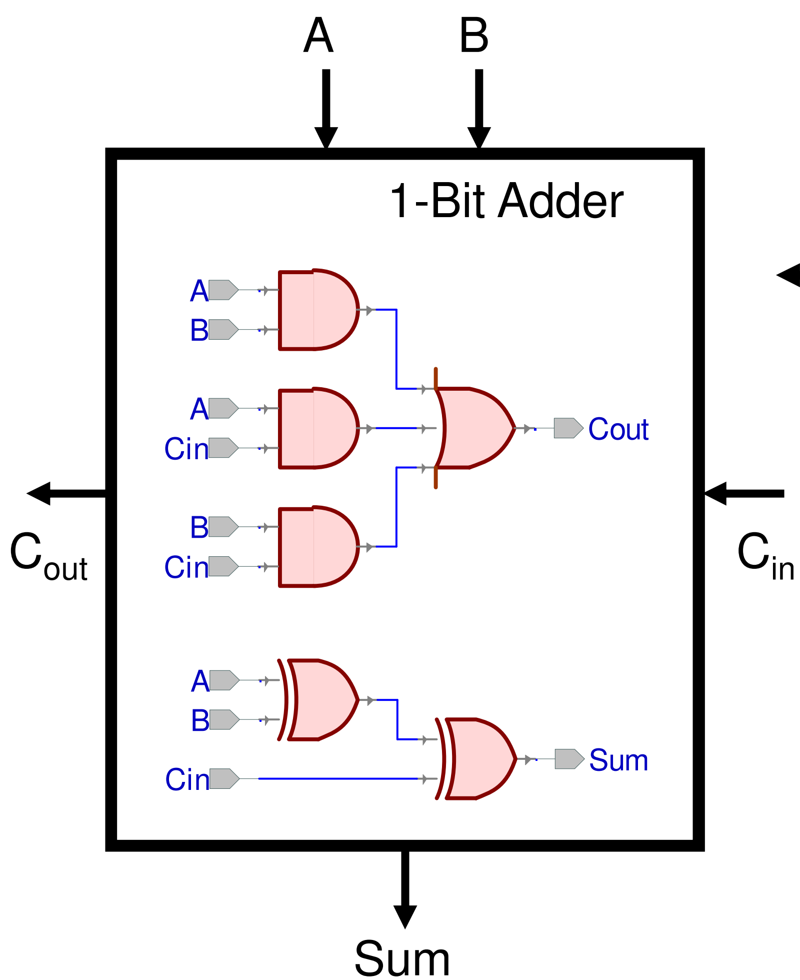
The theorems of Boolean algebra can simplify expressions

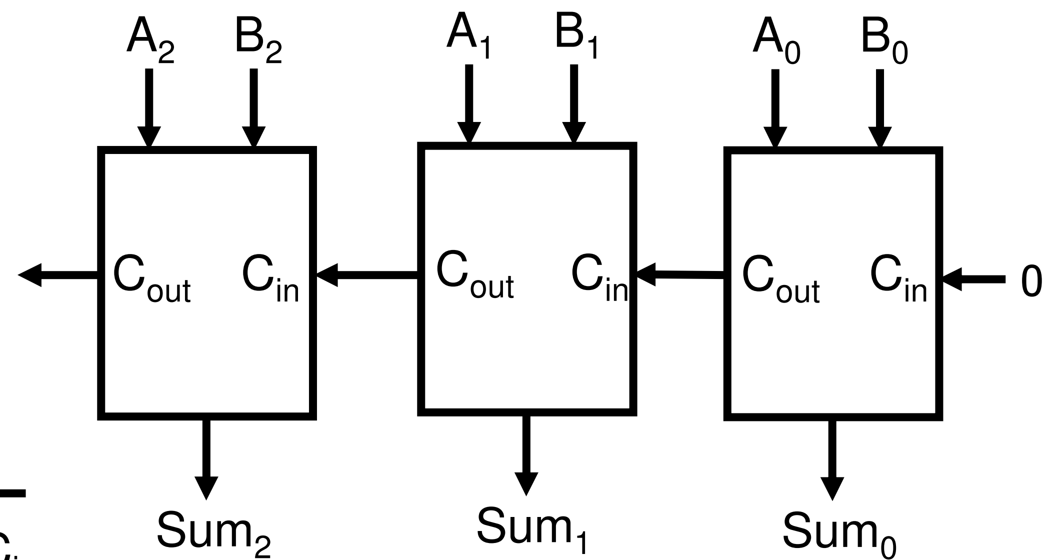
– **e.g., full adder's carry-out function**

$$
\begin{aligned}
\text{Cout} &= A'\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= A'\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + \boxed{A\,B\,Cin + A\,B\,Cin} \\
&= A'\,B\,Cin + A\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= (A' + A)\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= (1)\,B\,Cin + A\,B'\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,B'\,Cin + A\,B\,Cin' + \boxed{A\,B\,Cin + A\,B\,Cin} \\
&= B\,Cin + A\,B'\,Cin + A\,B\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,(B' + B)\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,(1)\,Cin + A\,B\,Cin' + A\,B\,Cin \\
&= B\,Cin + A\,Cin + A\,B\,(Cin' + Cin) \\
&= B\,Cin + A\,Cin + A\,B\,(1) \\
&= B\,Cin + A\,Cin + A\,B
\end{aligned}
$$

adding extra terms creates new factoring opportunities

Multi-bit Ripple-Carry Adder





Mapping Truth Tables to Logic Gates

Given a truth table:

1. **Write the Boolean expression**
2. **Minimize the Boolean expression**
3. **Draw as gates**
4. **Map to available gates**

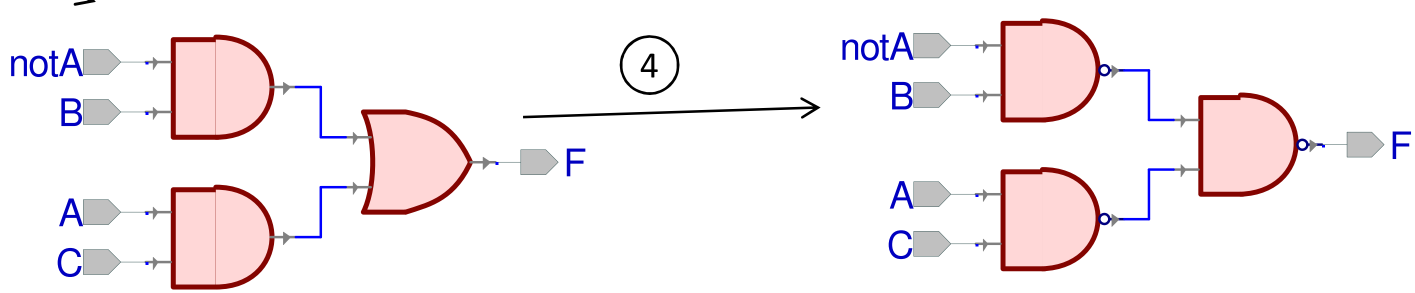
A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

1

$$
\begin{aligned}
F &= A'BC' + A'BC + AB'C + ABC \\
&= A'B(C' + C) + AC(B' + B) \\
&= A'B + AC
\end{aligned}
$$

2

3



Canonical Forms

- Truth table is the unique signature of a Boolean Function
- The same truth table can have many gate realizations
  - We've seen this already
  - Depends on how good we are at Boolean simplification
- Canonical forms
  - Standard forms for a Boolean expression
  - We all come up with the same expression

Sum-of-Products Canonical Form

- AKA **Disjunctive Normal Form (DNF)**
- AKA **Minterm Expansion**

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 **Read T rows off truth table**

2 **Convert to Boolean Algebra**

001 → A'B'C

011 → A'BC

101 → AB'C

110 → ABC'

111 → ABC

F

3 **Add the minterms together**

F= A'B'C + A'BC + AB'C + ABC' + ABC'

Sum-of-Products Canonical Form

Product term (or minterm)

- ANDed product of literals – input combination for which output is true
- each variable appears exactly once, true or inverted (but not both)

A	B	C	minterms
0	0	0	$A'B'C'$
0	0	1	$A'B'C$
0	1	0	$A'BC'$
0	1	1	$A'BC$
1	0	0	$AB'C'$
1	0	1	$AB'C$
1	1	0	ABC'
1	1	1	ABC

F in canonical form:

$$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$$

canonical form $\neq$ minimal form

$$\begin{aligned} F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\ &= (A'B' + A'B + AB' + AB)C + ABC' \\ &= ((A' + A)(B' + B))C + ABC' \\ &= C + ABC' \\ &= ABC' + C \\ &= AB + C \end{aligned}$$

Product-of-Sums Canonical Form

- AKA **Conjunctive Normal Form (CNF)**
- AKA **Maxterm Expansion**

④ Multiply the maxterms together

F =

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

① Read F rows off truth table

② Negate all bits

③ Convert to Boolean Algebra

F

Product-of-Sums Canonical Form

- AKA **Conjunctive Normal Form (CNF)**
- AKA **Maxterm Expansion**

④ Multiply the maxterms together

$$F = (A + B + C)(A + B' + C)(A' + B + C)$$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

① Read F rows off truth table	② Negate all bits	③ Convert to Boolean Algebra
000 →	111 →	$A + B + C$
010 →	101 →	$A + B' + C$
100 →	011 →	$A' + B + C$

F

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a minterm expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a minterm expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Taking the complement of both sides…

$$(F')' = (A'B'C' + A'BC' + AB'C')'$$

And using DeMorgan/Comp….

$$F = (A'B'C')' \ (A'BC')' \ (AB'C')'$$

$$F = (A + B + C)(A + B' + C)(A' + B + C)$$

Product-of-Sums Canonical Form

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	$A+B+C$
0	0	1	$A+B+C'$
0	1	0	$A+B'+C$
0	1	1	$A+B'+C'$
1	0	0	$A'+B+C$
1	0	1	$A'+B+C'$
1	1	0	$A'+B'+C$
1	1	1	$A'+B'+C'$

F in canonical form:

$$F(A, B, C) = (A + B + C)\,(A + B' + C)\,(A' + B + C)$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)\,(A + B' + C)\,(A' + B + C) \\ &= (A + B + C)\,(A + B' + C) \\ &\quad\ (A + B + C)\,(A' + B + C) \\ &= (A + C)\,(B + C) \end{aligned}$$