CSE 311: Foundations of Computing I

Section 6: Induction and Strong Induction Solutions

1. Induction

(a) Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid (n^3 + (n+1)^3 + (n+2)^3)$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid (2^3 + (2+1)^3 + (2+2)^3)$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid (k^3 + (k+1)^3 + (k+2)^3)$ for some arbitrary integer $k > 1$. Note that this is equivalent to assuming that $k^3 + (k+1)^3 + (k+2)^3 = 9\ell$ for some integer ℓ.

Induction Step: Goal: Show $9 \mid ((k+1)^3 + (k+2)^3 + (k+3)^3)$

$$\begin{aligned}
(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k+3)^3 + 9\ell - k^3 \text{ for some integer } \ell \quad \text{[Induction Hypothesis]} \\
&= k^3 + 9k^2 + 27k + 27 + 9\ell - k^3 \\
&= 9k^2 + 27k + 27 + 9\ell \\
&= 9(k^2 + 3k + 3 + \ell)
\end{aligned}$$

So $9 \mid ((k+1)^3 + (k+2)^3 + (k+3)^3)$, so $P(k) \to P(k+1)$ for an arbitrary integer $k > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

(b) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Solution:

Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction.

Base Case $(n = 6)$**:** $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Induction Hypothesis: Assume that $6j + 6 < 2^j$ for some arbitrary integer $j \geq 6$.

Induction Step: Goal: Show $6(j+1) + 6 < 2^{j+1}$

$$\begin{aligned}
6(j+1) + 6 &= 6j + 6 + 6 \\
&< 2^j + 6 && \text{[Induction Hypothesis]} \\
&< 2^j + 2^j && \text{[Since } 2^j > 6 \text{, since } j \geq 6\text{]} \\
&< 2 \cdot 2^j \\
&< 2^{j+1}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by induction.

(c) Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

Solution:

We define H_i more formally as $\sum_{k=1}^{i} \frac{1}{k}$. Let $P(n)$ be "$H_{2^n} \geq 1 + \frac{n}{2}$". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case $(n = 0)$**:** $H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$, so $P(0)$ holds.

Induction Hypothesis: Assume that $H_{2^j} \geq 1 + \frac{j}{2}$ for some arbitrary integer $j \in \mathbb{N}$.

Induction Step: Goal: Show $H_{2^{j+1}} \geq 1 + \dfrac{j+1}{2}$

$$\begin{aligned}
H_{2^{j+1}} &= \sum_{k=1}^{2^{j+1}} \frac{1}{k} \\
&= \sum_{k=1}^{2^j} \frac{1}{k} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\
&\geq 1 + \frac{j}{2} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} && \text{[Induction Hypothesis]} \\
&\geq 1 + \frac{j}{2} + 2^j \cdot \frac{1}{2^{j+1}} && \text{[There are } 2^j \text{ terms in } [2^j+1, 2^{j+1}] \text{ and each is at least } \tfrac{1}{2^{j+1}}] \\
&\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \\
&\geq 1 + \frac{j}{2} + \frac{1}{2} \geq 1 + \frac{j+1}{2}
\end{aligned}$$

So $P(j) \rightarrow P(j+1)$ for an arbitrary integer $j \in \mathbb{N}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

2. Strong Induction

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$\begin{aligned}
f(0) &= 0 \\
f(1) &= 1 \\
f(n) &= 2f(n-1) - f(n-2) \text{ for } n \geq 2
\end{aligned}$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n.

Solution:

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n.

Base Case $n = 0$**:** $f(0) = 0$ by definition.

Induction Hypothesis: Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every integer j with $0 \leq j \leq k$.

Induction Step: We show $P(k+1)$: We have two cases depending on whether $k+1 = 1$ or $k+1 \geq 2$. When $k+1 = 1$, we have $f(k+1) = f(1) = 1 = k+1$ which is what we needed to prove. When $k+1 \geq 2$, we have

$$
\begin{aligned}
f(k+1) &= 2f(k) - f(k-1) && \text{[Definition of } f\text{]} \\
&= 2k - (k-1) && \text{[Induction Hypothesis]} \\
&= k+1 && \text{[Algebra]}
\end{aligned}
$$

Therefore $P(k+1)$ is true in all cases.

Therefore, by induction $f(n) = n$ for all $n \in \mathbb{N}$.