Section 5: Number Theory and Induction

1. GCD
   (a) Calculate \( \gcd(100, 50) \).

   (b) Calculate \( \gcd(17, 31) \).

   (c) Find the multiplicative inverse of 6 modulo 7.

   (d) Does 49 have a multiplicative inverse modulo 7?

2. Extended Euclidean Algorithm
   (a) Find the multiplicative inverse \( y \) of 7 \(\pmod{33}\). That is, find \( y \) such that \( 7y \equiv 1 \pmod{33} \). You should use the extended Euclidean Algorithm. Your answer should be in the range \( 0 \leq y < 33 \).

   (b) Now, solve \( 7z \equiv 2 \pmod{33} \) for all of its integer solutions \( z \).

3. Induction
   (a) For any \( n \in \mathbb{N} \), define \( S_n \) to be the sum of the squares of the first \( n \) positive integers, or

   \[ S_n = 1^2 + 2^2 + \cdots + n^2. \]

   Prove that for all \( n \in \mathbb{N} \), \( S_n = \frac{1}{6}n(n+1)(2n+1) \).

   (b) Define the triangle numbers as \( \Delta_n = 1 + 2 + \cdots + n \), where \( n \in \mathbb{N} \). We showed in lecture that \( \Delta_n = \frac{n(n+1)}{2} \).

   Prove the following equality for all \( n \in \mathbb{N} \):

   \[ 0^3 + 1^3 + \cdots + n^3 = \Delta_n^2 \]

   (c) Prove for all \( n \in \mathbb{N} \) that if you have two groups of numbers, \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \), such that \( \forall (i \in [n]), a_i \leq b_i \), then it must be that:

   \[ a_1 + \cdots + a_n \leq b_1 + \cdots + b_n \]

4. Casting Out Nines
   (a) Suppose that \( a \equiv b \pmod{m} \). Prove by induction that for every integer \( n \geq 1 \), \( a^n \equiv b^n \pmod{m} \).

   (b) Let \( K \in \mathbb{N} \). Prove that if \( K \equiv 0 \pmod{9} \), then the sum of the digits of \( K \) is a multiple of 9.