Section 7: Structural Induction and Regular Expressions

1. Structural Induction

(a) Consider the following recursive definition of strings Σ^* over an alphabet Σ.

Basis Step: ϵ is a string

Recursive Step: If w is a string and $a \in \Sigma$ is a character then wa is a string.

Recall the following recursive definition of the function len:

$$\begin{aligned} \mathsf{len}(\epsilon) &= 0 \\ \mathsf{len}(wa) &= 1 + \mathsf{len}(w) \end{aligned}$$

Now, consider the following recursive definition:

$$\begin{aligned} \mathsf{double}(\epsilon) &= \epsilon \\ \mathsf{double}(wa) &= \mathsf{double}(w)aa. \end{aligned}$$

Prove that for any string x, $\mathsf{len}(\mathsf{double}(x)) = 2\mathsf{len}(x)$.

(b) Consider the following definition of a (rooted binary) **Tree**:

Basis Step: $\bullet$ is a **Tree**.

Recursive Step: If L is a **Tree** and R is a **Tree** then $\texttt{Tree}(\bullet, L, R)$ is a **Tree**.

The function leaves returns the number of leaves of a **Tree**. It is defined as follows:

$$\begin{aligned} \mathsf{leaves}(\bullet) &= 1 \\ \mathsf{leaves}(\texttt{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R) \end{aligned}$$

Also, recall the definition of size on trees:

$$\begin{aligned} |\bullet| &= 1 \\ |\texttt{Tree}(\bullet, L, R)| &= 1 + |L| + |R| \end{aligned}$$

Prove that $\mathsf{leaves}(T) \geq |T|/2 + 1/2$ for all Trees T.

2. Regular Expressions

(a) Write a regular expression that matches base 10 non-negative numbers (e.g., there should be no leading zeroes).

(b) Write a regular expression that matches all non-negative base-3 numbers that are divisible by 3.

(c) Write a regular expression that matches all binary strings that contain the substring "111", but not the substring "000".