Section 6: Induction and Strong Induction

1. Harmonic 9s

(a) Prove that $9 \mid n^3 + (n + 1)^3 + (n + 2)^3$ for all $n > 1$ by induction.

**Solution:**
Let $P(n)$ be “$9 \mid n^3 + (n + 1)^3 + (n + 2)^3$”. We will prove $P(n)$ for all integers $n > 1$ by induction.

**Base Case** ($n = 2$): $2^3 + (2 + 1)^3 + (2 + 2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2 + 1)^3 + (2 + 2)^3$, so $P(2)$ holds.

**Induction Hypothesis:** Assume that $9 \mid j^3 + (j + 1)^3 + (j + 2)^3$ for an arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j + 1)^3 + (j + 2)^3 = 9k$ for some integer $k$.

**Induction Step:** Goal: Show $9 \mid (j + 1)^3 + (j + 2)^3 + (j + 3)^3$

\[
\begin{align*}
(j + 1)^3 + (j + 2)^3 + (j + 3)^3 &= (j + 3)^3 + j^3 + (j + 1)^3 + (j + 2)^3 - j^3 \\
&= (j + 3)^3 + 9k - j^3 \quad \text{for some integer } k \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{align*}
\]

So $9 \mid (j + 1)^3 + (j + 2)^3 + (j + 3)^3$, so $P(j) \rightarrow P(j + 1)$ for an arbitrary integer $j > 1$.

**Conclusion:** $P(n)$ holds for all integers $n > 1$ by induction.

(b) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

**Solution:**
Let $P(n)$ be “$6n + 6 < 2^n$”. We will prove $P(n)$ for all integers $n \geq 6$ by induction.

**Base Case** ($n = 6$): $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

**Induction Hypothesis:** Assume that $6j + 6 < 2^j$ for an arbitrary integer $j \geq 6$.

**Induction Step:** Goal: Show $6(j + 1) + 6 < 2^{j+1}$

\[
\begin{align*}
6(j + 1) + 6 &= 6j + 6 + 6 \\
&< 2^j + 6 \quad \text{[Induction Hypothesis]} \\
&< 2^j + 2^j \quad [\text{Since } 2^j > 6, \text{ since } j \geq 6] \\
&< 2 \cdot 2^j \\
&< 2^{j+1}
\end{align*}
\]

So $P(j) \rightarrow P(j + 1)$ for an arbitrary integer $j \geq 6$.

**Conclusion:** $P(n)$ holds for all integers $n \geq 6$ by induction.

(c) Define

\[H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}\]
Prove that $H_{2n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

**Solution:**

We define $H_1$ more formally as $\sum_{k=1}^{i} \frac{1}{k}$. Let $P(n)$ be “$H_{2n} \geq 1 + \frac{n}{2}$”. We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

**Base Case** ($n = 0$): $H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$, so $P(0)$ holds.

**Induction Hypothesis:** Assume that $H_{2j} \geq 1 + \frac{j}{2}$ for an arbitrary integer $j \in \mathbb{N}$.

**Induction Step:** Goal: Show $H_{2j+1} \geq 1 + \frac{j+1}{2}$

\[
H_{2j+1} = \sum_{k=1}^{2j+1} \frac{1}{k} \\
= \sum_{k=1}^{2j} \frac{1}{k} + \sum_{k=2j+1}^{2j+1} \frac{1}{k} \\
\geq 1 + \frac{j}{2} + \sum_{k=2j+1}^{2j+1} \frac{1}{k} \\
\geq 1 + \frac{j}{2} + \frac{1}{2j+1} \quad \text{[Induction Hypothesis]} \\
\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \quad \text{[There are $2^j$ terms in $[2^j+1,2^{j+1}]$ and each is at least $\frac{1}{2^{j+1}}$]} \\
\geq 1 + \frac{j}{2} + \frac{1}{2} \geq 1 + \frac{j+1}{2}
\]

So $P(j) \rightarrow P(j+1)$ for an arbitrary integer $j \in \mathbb{N}$.

**Conclusion:** $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

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2. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 4 or 5 at a time (every dog gets walked exactly once). Prove the dog walker can always split the $n$ dogs into groups of 4 or 5.

**Solution:**

Let $P(n)$ be “a group with $n$ dogs can be split into groups of 4 or 5 dogs.” We will prove $P(n)$ for all natural numbers $n \geq 12$ by strong induction.

**Base Case** $n = 12, 13, 14, \text{ or } 15$: $12 = 4 + 4 + 4, 13 = 4 + 4 + 5, 14 = 4 + 5 + 5, 15 = 5 + 5 + 5$. So $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.

**Induction Hypothesis:** Assume that $P(12), \ldots, P(n)$ hold for $n \geq 15$.

**Induction Step:** Goal: Show $n+1$ dogs can be split into groups of size 4 or 5. We first form one group of 4 dogs. Then we can divide the remaining $n-3$ dogs into groups of 4 or 5 by the assumption $P(n-3)$. (Note that $n \geq 15$ and so $n-3 \geq 12$; thus, $P(n-3)$ is among our assumptions $P(12), \ldots, P(n)$.)

**Conclusion:** $P(n)$ holds for all integers $n \geq 12$ by strong induction.
3. Cantelli’s rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function $f$:

$$
\begin{align*}
f(0) &= 0 \\
f(1) &= 1 \\
f(n) &= 2f(n - 1) - f(n - 2) \text{ for } n \geq 2
\end{align*}
$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year $n$.

**Solution:**

Let $P(n)$ be “$f(n) = n$”. We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on $n$.

**Base Cases** ($n = 0, n = 1$): $f(0) = 0$ and $f(1) = 1$ by definition.

**Induction Hypothesis:** Assume that $P(0) \land P(1) \land \ldots P(n - 1)$ are true for some fixed but arbitrary $n - 1 \geq 1$.

**Induction Step:** We show $P(n)$:

$$
\begin{align*}
f(n) &= 2f(n - 1) - f(n - 2) & \text{Definition of } f \\
      &= 2(n - 1) - (n - 2) & \text{Induction Hypothesis on } P(n - 1) \text{ and } P(n - 2) \\
      &= n & \text{Algebra}
\end{align*}
$$

Note that we have indeed assumed $P(n - 1) \land P(n - 2)$ because $n \geq 2$ and we showed base cases $P(0)$ and $P(1)$.

**Conclusion:** Therefore, by strong induction $P(n)$ is true for all $n \in \mathbb{N}$.