# Section 6: Induction and Strong Induction

## 1. Harmonic 9s

(a) Prove that $9 \mid n^3 + (n+1)^3 + (n+2)^3$ for all $n > 1$ by induction.

**Solution:**

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

**Base Case** ($n = 2$)**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

**Induction Hypothesis:** Assume that $9 \mid j^3 + (j+1)^3 + (j+2)^3$ for an arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer $k$.

**Induction Step:** $\boxed{\text{Goal: Show } 9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3}$

$$\begin{aligned}
(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + j^3 + (j+1)^3 + (j+2)^3 - j^3 && \text{Rearrange, add and subtract } j^3 \\
&= (j+3)^3 + 9k - j^3 \text{ for some integer } k && \text{[Induction Hypothesis]} \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}$$

So $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for an arbitrary integer $j > 1$.

**Conclusion:** $P(n)$ holds for all integers $n > 1$ by induction.

(b) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

**Solution:**

Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction.

**Base Case** ($n = 6$)**:** $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

**Induction Hypothesis:** Assume that $6j + 6 < 2^j$ for an arbitrary integer $j \geq 6$.

**Induction Step:** $\boxed{\text{Goal: Show } 6(j+1) + 6 < 2^{j+1}}$

$$\begin{aligned}
6(j+1) + 6 &= 6j + 6 + 6 \\
&< 2^j + 6 && \text{[Induction Hypothesis]} \\
&< 2^j + 2^j && \text{[Since } 2^j > 6\text{, since } j \geq 6\text{]} \\
&< 2 \cdot 2^j \\
&< 2^{j+1}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \geq 6$.

**Conclusion:** $P(n)$ holds for all integers $n \geq 6$ by induction.

(c) Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

**Solution:**

We define $H_i$ more formally as $\sum_{k=1}^{i} \frac{1}{k}$. Let $P(n)$ be "$H_{2^n} \geq 1 + \frac{n}{2}$". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

**Base Case** $(n = 0)$**:** $H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$, so $P(0)$ holds.

**Induction Hypothesis:** Assume that $H_{2^j} \geq 1 + \frac{j}{2}$ for an arbitrary integer $j \in \mathbb{N}$.

**Induction Step:** $\boxed{\text{Goal: Show } H_{2^{j+1}} \geq 1 + \frac{j+1}{2}}$

$$
\begin{aligned}
H_{2^{j+1}} &= \sum_{k=1}^{2^{j+1}} \frac{1}{k} \\
&= \sum_{k=1}^{2^j} \frac{1}{k} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\
&\geq 1 + \frac{j}{2} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} && \text{[Induction Hypothesis]} \\
&\geq 1 + \frac{j}{2} + 2^j \cdot \frac{1}{2^{j+1}} && \text{[There are } 2^j \text{ terms in } [2^j + 1, 2^{j+1}] \text{ and each is at least } \tfrac{1}{2^{j+1}}] \\
&\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \\
&\geq 1 + \frac{j}{2} + \frac{1}{2} \geq 1 + \frac{j+1}{2}
\end{aligned}
$$

So $P(j) \to P(j + 1)$ for an arbitrary integer $j \in \mathbb{N}$.

**Conclusion:** $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

## 2. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 4 or 5 at a time (every dog gets walked exactly once). Prove the dog walker can always split the $n$ dogs into groups of 4 or 5.

**Solution:**

Let $P(n)$ be "a group with $n$ dogs can be split into groups of 4 or 5 dogs." We will prove $P(n)$ for all natural numbers $n \geq 12$ by strong induction.

**Base Case** $n = 12, 13, 14,$ **or** $15$**:** $12 = 4 + 4 + 4, 13 = 4 + 4 + 5, 14 = 4 + 5 + 5, 15 = 5 + 5 + 5$. So $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.

**Induction Hypothesis:** Assume that $P(12), \ldots, P(n)$ hold for $n \geq 15$.

**Induction Step:** $\boxed{\text{Goal: Show n+1 dogs can be split into groups of size 4 or 5.}}$ We first form one group of 4 dogs. Then we can divide the remaining $n-3$ dogs into groups of 4 or 5 by the assumption $P(n-3)$. (Note that $n \geq 15$ and so $n - 3 \geq 12$; thus, $P(n - 3)$ is among our assumptions $P(12), \ldots, P(n)$.)

**Conclusion:** $P(n)$ holds for all integers $n \geq 12$ by strong induction.

## 3. Cantelli's rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function $f$:

$$\begin{aligned} &f(0) = 0 \\ &f(1) = 1 \\ &f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2 \end{aligned}$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year $n$.

**Solution:**

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on $n$.

**Base Cases** $(n = 0, n = 1)$**:** $f(0) = 0$ and $f(1) = 1$ by definition.

**Induction Hypothesis:** Assume that $P(0) \wedge P(1) \wedge \ldots P(n-1)$ are true for some fixed but arbitrary $n - 1 \geq 1$.

**Induction Step:** We show $P(n)$:

$$\begin{aligned} f(n) &= 2f(n-1) - f(n-2) && \text{Definition of } f \\ &= 2(n-1) - (n-2) && \text{Induction Hypothesis on } P(n-1) \text{ and } P(n-2) \\ &= n && \text{Algebra} \end{aligned}$$

Note that we have indeed assumed $P(n-1) \wedge P(n-2)$ because $n \geq 2$ and we showed base cases $P(0)$ and $P(1)$.

**Conclusion:** Therefore, by strong induction $P(n)$ is true for all $n \in \mathbb{N}$.