Section 3: Inference

1. Using the Direct Proof Rule

Show that $\neg p \to s$ follows from $p \vee q$, $q \to r$ and $r \to s$.

Solution:

1.	$p \vee q$		[Given]
2.	$q \to r$		[Given]
3.	$r \to s$		[Given]
	4.1. $\neg p$	[Assumption]	
	4.2. q	[Elim of $\vee$: 1, 4.1]	
	4.3. r	[MP of 4.2, 2]	
	4.4. s	[MP 4.3, 3]	
4.	$\neg p \to s$		[Direct Proof Rule]

2. A Formal Proof in Propositional Logic

Show that $\neg p$ follows from $\neg(\neg r \vee t)$, $\neg q \vee \neg s$ and $(p \to q) \wedge (r \to s)$.

Solution:

1.	$\neg(\neg r \vee t)$	[Given]
2.	$\neg q \vee \neg s$	[Given]
3.	$(p \to q) \wedge (r \to s)$	[Given]
4.	$\neg\neg r \wedge \neg t$	[DeMorgan's Law: 1]
5.	$\neg\neg r$	[Elim of $\wedge$: 4]
6.	r	[Double Negation: 5]
7.	$r \to s$	[Elim of $\wedge$: 3]
8.	s	[MP, 6,7]
9.	$\neg\neg s$	[Double Negation: 8]
10.	$\neg s \vee \neg q$	[Commutative: 2]
11.	$\neg q$	[Elim of $\vee$: 10, 9]
12.	$p \to q$	[Elim of $\wedge$: 3]
13.	$\neg q \to \neg p$	[Contrapositive: 12]
14.	$\neg p$	[MP: 11,13]

3. A Formal Proof in Predicate Logic

Prove $\exists x\, (P(x) \vee R(x))$ from $\forall x\, (P(x) \vee Q(x))$ and $\forall y\, (\neg Q(y) \vee R(y))$.

Solution:

1.	$\forall x\ (P(x) \lor Q(x))$	[Given]
2.	$\forall y\ (\neg Q(y) \lor R(y))$	[Given]
3.	$P(a) \lor Q(a)$	[Elim $\forall$: 1]
4.	$\neg Q(a) \lor R(a)$	[Elim $\forall$: 2]
5.	$Q(a) \to R(a)$	[Law of Implication: 4]
6.	$\neg\neg P(a) \lor Q(a)$	[Double Negation: 3]
7.	$\neg P(a) \to Q(a)$	[Law of Implication: 5]
	8.1. $\neg P(a)$ [Assumption]	
	8.2. $Q(a)$ [Modus Ponens: 8.1, 7]	
	8.3. $R(a)$ [Modus Ponens: 8.2, 5]	
8.	$\neg P(a) \to R(a)$	[Direct Proof]
9.	$\neg\neg P(a) \lor R(a)$	[Law of Implication: 8]
10.	$P(a) \lor R(a)$	[Double Negation: 9]
11.	$\exists x\ (P(x) \lor R(x))$	[Intro $\exists$: 10]

4. Formal Spoofs

For each of of the following proofs, determine why the proof is incorrect. Then, show that the claim is true by fixing the error.

(a) Show that $p \to (q \lor r)$ follows from $p \to q$ and r.

1.	$p \to q$	[Given]
2.	r	[Given]
3.	$p \to (q \lor r)$	[$\lor$ Intro: 1, 2]

(b) Show that q follows from $\neg p \lor q$ and p.

1.	$\neg p \lor q$	[Given]
2.	p	[Given]
3.	q	[$\lor$ Elim: 1, 2]

(c) Show that q follows from $q \lor p$ and $\neg p$.

1.	$\neg p$	[Given]
2.	$q \lor p$	[Given]
3.	$q \lor \mathsf{F}$	[Substitute $p = \mathsf{F}$ since $\neg p$ holds: 1, 2]
4.	q	[Identity: 3]

Solution:

The mistakes are as follows:

(a) Line 3: inference rule used on a subexpression.

(b) Line 3: $\lor$ Elim requires $\neg\neg p$ not p.

(c) Line 3: there is no such "substitute for" rule

Next, we consider how to fix the proofs:

(a) Since r is true, $q \vee r$ is true. Hence, the latter is true if we assume p. (It's true even if we don't assume p.) This can be formalized using the direct proof rule.

(b) Add a line inferring $\neg\neg p$ from p.

(c) Line 4 follows instead by $\vee$ Elim.