# Section 2: Equivalences and Boolean Algebra

## 1. Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) $\neg p \to (q \to r)$ $\qquad$ $q \to (p \lor r)$

(b) $p \leftrightarrow q$ $\qquad$ $(p \land q) \lor (\neg p \land \neg q)$

## 2. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) $\neg p \lor (\neg q \lor (p \land q))$

(b) $\neg(p \lor (q \land p))$

## 3. Translating Between Predicate Logic and English

Let the domain of discourse be integers. Let's define the predicates $\mathsf{Even}(x)$ and $\mathsf{Odd}(x)$ to mean that $x$ is an even or odd number, respectively. Define the predicates $\mathsf{Positive}(x)$, $\mathsf{Negative}(x)$, and $\mathsf{Prime}(x)$ to mean that $x$ is positive, negative, or prime, respectively.

Translate the logical statements into English, and translate the English statements into predicate logic. You should not simplify. However, you should use the techniques shown in lecture for producing more natural translations when restricting domains and for avoiding the introduction of variable names when not necessary. You can also assume an "$<$" operator that is true when $x < y$.

(a) $\forall x.(\mathsf{Even}(x) \oplus \mathsf{Odd}(x))$

(b) $\exists x.\neg(\mathsf{Negative}(x) \lor \mathsf{Positive}(x))$

(c) There exists an even prime integer.

(d) There are infinitely many prime integers.

## 4. Properties of XOR

Like $\land$ and $\lor$, the $\oplus$ operator (exclusive or) has many interesting properties. For example, it is easy to verify with a truth table that $\oplus$ is also associative. In this problem, we will prove some additional properties of $\oplus$.

Use equivalence chains to prove each of the facts stated below. For this problem only, you may also use the equivalence

$$p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$$

which you may cite as "Definition of $\oplus$". This equivalence allows you to translate $\oplus$ into an expression involving only $\land$, $\lor$, and $\neg$, so that the standard equivalences can then be applied.

(a) $p \oplus q \equiv q \oplus p$ (Commutativity)

(b) $p \oplus p \equiv \mathsf{F}$ and $p \oplus \neg p \equiv \mathsf{T}$

(c) $p \oplus \mathsf{F} \equiv p$ and $p \oplus \mathsf{T} \equiv \neg p$

(d) $(\neg p) \oplus q \equiv \neg(p \oplus q) \equiv p \oplus (\neg q)$. I.e., negating one of the inputs negates the overall expression.

## 5. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) $p \to q$ $\qquad$ $q \to p$

(b) $p \to (q \land r)$ $\qquad$ $(p \to q) \land r$