Section 2: Equivalences and Boolean Algebra

1. Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) $\neg p \to (q \to r)$ $\quad$ $q \to (p \lor r)$

Solution:

$$
\begin{aligned}
\neg p \to (q \to r) &\equiv \neg\neg p \lor (q \to r) && \text{[Law of Implication]}\\
&\equiv p \lor (q \to r) && \text{[Double Negation]}\\
&\equiv p \lor (\neg q \lor r) && \text{[Law of Implication]}\\
&\equiv (p \lor \neg q) \lor r && \text{[Associativity]}\\
&\equiv (\neg q \lor p) \lor r && \text{[Commutativity]}\\
&\equiv \neg q \lor (p \lor r) && \text{[Associativity]}\\
&\equiv q \to (p \lor r) && \text{[Law of Implication]}
\end{aligned}
$$

(b) $p \leftrightarrow q$ $\quad$ $(p \land q) \lor (\neg p \land \neg q)$

Solution:

$$
\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \land (q \to p) && \text{[iff is two implications]}\\
&\equiv (\neg p \lor q) \land (q \to p) && \text{[Law of Implication]}\\
&\equiv (\neg p \lor q) \land (\neg q \lor p) && \text{[Law of Implication]}\\
&\equiv ((\neg p \lor q) \land \neg q) \lor ((\neg p \lor q) \land p) && \text{[Distributivity]}\\
&\equiv (\neg q \land (\neg p \lor q)) \lor ((\neg p \lor q) \land p) && \text{[Commutativity]}\\
&\equiv ((\neg q \land \neg p) \lor (\neg q \land q)) \lor ((\neg p \lor q) \land p) && \text{[Distributivity]}\\
&\equiv ((\neg p \land \neg q) \lor (\neg q \land q)) \lor ((\neg p \lor q) \land p) && \text{[Commutativity]}\\
&\equiv ((\neg p \land \neg q) \lor (q \land \neg q)) \lor ((\neg p \lor q) \land p) && \text{[Commutativity]}\\
&\equiv ((\neg p \land \neg q) \lor (q \land \neg q)) \lor (p \land (\neg p \lor q)) && \text{[Commutativity]}\\
&\equiv ((\neg p \land \neg q) \lor (q \land \neg q)) \lor ((p \land \neg p) \lor (p \land q)) && \text{[Distributivity]}\\
&\equiv ((\neg p \land \neg q) \lor F) \lor ((p \land \neg p) \lor (p \land q)) && \text{[Negation]}\\
&\equiv ((\neg p \land \neg q) \lor F) \lor (F \lor (p \land q)) && \text{[Negation]}\\
&\equiv (\neg p \land \neg q) \lor (F \lor (p \land q)) && \text{[Identity]}\\
&\equiv (\neg p \land \neg q) \lor ((p \land q) \lor F) && \text{[Commutativity]}\\
&\equiv (\neg p \land \neg q) \lor (p \land q) && \text{[Identity]}\\
&\equiv (p \land q) \lor (\neg p \land \neg q) && \text{[Commutativity]}
\end{aligned}
$$

2. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) $\neg p \vee (\neg q \vee (p \wedge q))$

Solution:

A: First, we replace $\neg$, $\vee$, and $\wedge$. This gives us $p' + q' + pq$. (Note that the parentheses are not necessary in boolean algebra since the operations are associative.) Next, we can use DeMorgan's laws to get the slightly simpler $(pq)' + pq$. Then, we can use commutativity to get $pq + (pq)'$ and complementarity to get 1. (Note that this is another way of saying the formula is a tautology.)

B: Replacing $\neg$, $\vee$, and $\wedge$ gives us $p' + q' + pq$. (Note that parentheses are not necessary in boolean algebra since the operations are associative.) Then, we can simplify as follows:

$$\begin{aligned} p' + q' + pq &= (pq)' + pq && \text{De Morgan} \\ &= pq + (pq)' && \text{Commutativity} \\ &= 1 && \text{Complementarity} \end{aligned}$$

Since 1 in boolean algebra means T, this shows that the original expression is a tautology.

(b) $\neg(p \vee (q \wedge p))$

Solution:

A:

$$\begin{aligned} (p + qp)' &= p'(qp)' && \text{De Morgan} \\ &= p'(q' + p') && \text{De Morgan (on second term)} \\ &= p'(p' + q') && \text{Commutativity} \\ &= p' && \text{Absorption} \end{aligned}$$

B: Translating to Boolean algebra, we get $(p + qp)'$. Then, we can simplify as follows:

$$\begin{aligned} (p + qp)' &= p'(qp)' && \text{De Morgan} \\ &= p'(q' + p') && \text{De Morgan} \\ &= p'(p' + q') && \text{Commutativity} \\ &= p' && \text{Absorption} \end{aligned}$$

3. Translating Between Predicate Logic and English

Let the domain of discourse be integers. Let's define the predicates $\mathsf{Even}(x)$ and $\mathsf{Odd}(x)$ to mean that x is an even or odd number, respectively. Define the predicates $\mathsf{Positive}(x)$, $\mathsf{Negative}(x)$, and $\mathsf{Prime}(x)$ to mean that x is positive, negative, or prime, respectively.

Translate the logical statements into English, and translate the English statements into predicate logic. You should not simplify. However, you should use the techniques shown in lecture for producing more natural translations when restricting domains and for avoiding the introduction of variable names when not necessary. You can also assume an "<" operator that is true when $x < y$.

(a) $\forall x.(\mathsf{Even}(x) \oplus \mathsf{Odd}(x))$

Solution:

Every integer is either odd or even.

(b) $\exists x.\neg(\mathsf{Negative}(x) \vee \mathsf{Positive}(x))$

Solution:

There exists an integer which is not positive or negative.

(c) There exists an even prime integer.

Solution:

$\exists x.(\mathsf{Even}(x) \land \mathsf{Prime}(x))$

(d) There are infinitely many prime integers.

Solution:

$\forall x.\exists y.(\mathsf{Prime}(x) \to (\mathsf{Prime}(y) \land (x < y)))$

4. Properties of XOR

Like $\land$ and $\lor$, the $\oplus$ operator (exclusive or) has many interesting properties. For example, it is easy to verify with a truth table that $\oplus$ is also associative. In this problem, we will prove some additional properties of $\oplus$.

Use equivalence chains to prove each of the facts stated below. For this problem only, you may also use the equivalence

$$p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$$

which you may cite as "Definition of $\oplus$". This equivalence allows you to translate $\oplus$ into an expression involving only $\land$, $\lor$, and $\neg$, so that the standard equivalences can then be applied.

(a) $p \oplus q \equiv q \oplus p$ (Commutativity)

Solution:

$$\begin{aligned}
p \oplus q &\equiv (p \land \neg q) \lor (\neg p \land q) && \text{Definition of } \oplus \\
&\equiv (\neg p \land q) \lor (p \land \neg q) && \text{Commutativity} \\
&\equiv (q \land \neg p) \lor (\neg q \land p) && \text{Commutativity} \\
&\equiv q \oplus p && \text{Definition of } \oplus
\end{aligned}$$

(b) $p \oplus p \equiv \mathsf{F}$ and $p \oplus \neg p \equiv \mathsf{T}$

Solution:

$$\begin{aligned}
p \oplus p &\equiv (p \land \neg p) \lor (\neg p \land p) && \text{Definition of } \oplus \\
&\equiv (p \land \neg p) \lor (p \land \neg p) && \text{Commutativity} \\
&\equiv (p \land \neg p) && \text{Idempotency} \\
&\equiv \mathsf{F} && \text{Negation}
\end{aligned}$$

$$\begin{aligned}
p \oplus \neg p &\equiv (p \land \neg\neg p) \lor (\neg p \land \neg p) && \text{Definition of } \oplus \\
&\equiv (p \land p) \lor (\neg p \land \neg p) && \text{Double Negation} \\
&\equiv p \lor \neg p && \text{Idempotency} \\
&\equiv \mathsf{T} && \text{Negation}
\end{aligned}$$

(c) $p \oplus \mathsf{F} \equiv p$ and $p \oplus \mathsf{T} \equiv \neg p$

Solution:

$$
\begin{aligned}
p \oplus \mathsf{F} &\equiv (p \wedge \neg \mathsf{F}) \vee (\neg p \wedge \mathsf{F}) && \text{Definition of } \oplus \\
&\equiv (p \wedge (\neg \mathsf{F} \vee \mathsf{F})) \vee (\neg p \wedge \mathsf{F}) && \text{Identity} \\
&\equiv (p \wedge (\mathsf{F} \vee \neg \mathsf{F})) \vee (\neg p \wedge \mathsf{F}) && \text{Commutativity} \\
&\equiv (p \wedge \mathsf{T}) \vee (\neg p \wedge \mathsf{F}) && \text{Negation} \\
&\equiv p \vee (\neg p \wedge \mathsf{F}) && \text{Identity} \\
&\equiv p \vee \mathsf{F} && \text{Domination} \\
&\equiv p && \text{Identity}
\end{aligned}
$$

$$
\begin{aligned}
p \oplus \mathsf{T} &\equiv (p \wedge \neg \mathsf{T}) \vee (\neg p \wedge \mathsf{T}) && \text{Definition of } \oplus \\
&\equiv (p \wedge \neg \mathsf{T}) \vee \neg p && \text{Identity} \\
&\equiv (\neg\neg p \wedge \neg \mathsf{T}) \vee \neg p && \text{Double Negation} \\
&\equiv \neg(\neg p \vee \mathsf{T}) \vee \neg p && \text{De Morgan} \\
&\equiv \neg \mathsf{T} \vee \neg p && \text{Domination} \\
&\equiv \neg(\mathsf{T} \wedge p) && \text{De Morgan} \\
&\equiv \neg(p \wedge \mathsf{T}) && \text{Commutativity} \\
&\equiv \neg p && \text{Identity}
\end{aligned}
$$

(d) $(\neg p) \oplus q \equiv \neg(p \oplus q) \equiv p \oplus (\neg q)$. I.e., negating one of the inputs negates the overall expression.

Solution:

$$
\begin{aligned}
\neg(p \oplus q) &\equiv \neg((p \wedge \neg q) \vee (\neg p \wedge q)) && \text{Definition of } \oplus \\
&\equiv \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) && \text{De Morgan} \\
&\equiv (\neg p \vee \neg\neg q) \wedge (\neg\neg p \vee \neg q) && \text{De Morgan} \\
&\equiv (\neg p \vee q) \wedge (p \vee \neg q) && \text{Double Negation} \\
&\equiv ((\neg p \vee q) \wedge p) \vee ((\neg p \vee q) \wedge \neg q) && \text{Distributivity} \\
&\equiv (p \wedge (\neg p \vee q)) \vee (\neg q \wedge (\neg p \vee q)) && \text{Commutativity} \\
&\equiv ((p \wedge \neg p) \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) && \text{Distributivity} \\
&\equiv ((p \wedge \neg p) \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee (q \wedge \neg q)) && \text{Commutativity} \\
&\equiv (\mathsf{F} \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee \mathsf{F}) && \text{Negation} \\
&\equiv ((p \wedge q) \vee \mathsf{F}) \vee ((\neg q \wedge \neg p) \vee \mathsf{F}) && \text{Commutativity} \\
&\equiv (p \wedge q) \vee (\neg q \wedge \neg p) && \text{Identity} \\
&\equiv (\neg\neg p \wedge q) \vee (\neg q \wedge \neg p) && \text{Double Negation} \\
&\equiv (\neg\neg p \wedge q) \vee (\neg p \wedge \neg q) && \text{Commutativity} \\
&\equiv (\neg p \wedge \neg q) \vee (\neg\neg p \wedge q) && \text{Commutativity} \\
&\equiv \neg p \oplus q && \text{Definition of } \oplus
\end{aligned}
$$

The second equivalence $\neg(p \oplus q) \equiv p \oplus (\neg q)$ follows from the first and Commutativity (part a).

5. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) $p \to q$ $\qquad$ $q \to p$

Solution:

A: They differ when $p = \mathsf{T}$ and $q = \mathsf{F}$.
B: They differ when $p = \mathsf{T}$ and $q = \mathsf{F}$ since $p \to q = \mathsf{T} \to \mathsf{F} \equiv \mathsf{F}$ but $q \to p \equiv \mathsf{F} \to \mathsf{T} \equiv \mathsf{T}$.

(b) $p \to (q \wedge r)$ $\qquad$ $(p \to q) \wedge r$

Solution:

A: They differ when $p = r = \mathsf{F}$ since $p \to (q \wedge r) = \mathsf{F} \to (q \wedge \mathsf{F}) \equiv \mathsf{T}$ but $(p \to q) \wedge r = (\mathsf{F} \to q) \wedge \mathsf{F} \equiv \mathsf{F}$.
B: They differ when $p = r = \mathsf{F}$ since $p \to (q \wedge r) = \mathsf{F} \to (q \wedge \mathsf{F}) \equiv \mathsf{F} \to \mathsf{F} \equiv \mathsf{T}$ and on the other hand $(p \to q) \wedge r = (\mathsf{F} \to q) \wedge \mathsf{F} \equiv \mathsf{T} \wedge \mathsf{F} \equiv \mathsf{F}$.