CSE 311 Lecture 28: Undecidability of the Halting Problem

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Topics

Countability and uncomputability
   A quick recap of Lecture 27.

Undecidability of the halting problem
   Important problems computers can’t solve.
Countability and uncomputability

A quick recap of Lecture 27.
Countable and uncountable sets

Countable set

A set is *countable* iff it has the same cardinality as some subset of $\mathbb{N}$.
Countable and uncountable sets

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Countable sets
- $\mathbb{N}$ (natural numbers)
- $\mathbb{Z}$ (integers)
- $\mathbb{Q}^+$ (positive rational numbers)
- $\Sigma^*$ over finite $\Sigma$
- All (Java) programs

Shown by dovetailing.
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- $\Sigma^*$ over finite $\Sigma$
- All (Java) programs

Uncountable sets
- All real numbers in $[0, 1)$

Shown by dovetailing.

Shown by diagonalization.
Recall the proof that $[0, 1)$ is uncountable: diagonalization

Suppose for contradiction that there is a list $\{r_0, r_1, r_2, \ldots\}$ of all real numbers in $[0, 1)$.

Consider the digits $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., the $n$-th digit of $r_n$ for $n \in \mathbb{N}$.

For each such digit $x_i$, construct the digit $\hat{x}_i$ as follows:

- If $x_i = 1$ then $\hat{x}_i = 0$.
- If $x_i \neq 1$ then $\hat{x}_i = 1$.

Now, consider the number $\hat{r} = 0.\hat{x}_0\hat{x}_1\hat{x}_2\hat{x}_3\ldots$

Note that $r_n \neq \hat{r}$ for any $n \in \mathbb{N}$ because they differ on the $n$-th digit.

So the list doesn’t include $\hat{r}$, which is a contradiction. Thus the set $[0, 1)$ is uncountable.
The set of all functions $f: \mathbb{N} \rightarrow \{0, 1\}$ is uncountable.
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Suppose for contradiction that there is a list $\{f_1, f_2, f_3, \ldots\}$ of functions from $\mathbb{N}$ to $\{0, 1\}$.

\[
\begin{align*}
  f_0 &\ 0000000000\ldots \\
  f_1 &\ 1111111111\ldots \\
  f_2 &\ 0101010101\ldots \\
  f_3 &\ 0111011110\ldots \\
  f_4 &\ 1100011001\ldots \\
  \vdots 
\end{align*}
\]
The set of all functions $f: \mathbb{N} \to \{0, 1\}$ is uncountable

Suppose for contradiction that there is a list \{\(f_1, f_2, f_3, \ldots\)\} of functions from \(\mathbb{N}\) to \{0, 1\}.

Consider the outputs \(x_0, x_1, x_2, x_3, \ldots\) on the diagonal of this list, i.e., \(f_n(n)\) for \(n \in \mathbb{N}\).

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\begin{align*}
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Consider the outputs $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., $f_n(n)$ for $n \in \mathbb{N}$.

For each such output $x_i$, construct $\hat{x}_i$ as follows:

- If $x_i = 1$ then $\hat{x}_i = 0$.
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For each such output \( x_i \), construct \( \hat{x}_i \) as follows:

- If \( x_i = 1 \) then \( \hat{x}_i = 0 \).
- If \( x_i \neq 1 \) then \( \hat{x}_i = 1 \).

Now, consider the function \( \hat{f}(n) = \hat{x}_n \).
The set of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable

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Now, consider the function $\hat{f}(n) = \hat{x}_n$.

Note that $f_n \neq \hat{f}$ for any $n \in \mathbb{N}$ because the functions differ on the $n$-th output.
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Note that $f_n \neq \hat{f}$ for any $n \in \mathbb{N}$ because the functions differ on the $n$-th output.

So the list doesn’t include $\hat{f}$, which is a contradiction. Thus the set $\{f \mid f: \mathbb{N} \to \{0, 1\}\}$ is uncountable.
Uncomputable functions

We have seen that …

- The set of all (Java) programs is countable.
- The set of all functions $f: \mathbb{N} \rightarrow \{0, 1\}$ is uncountable.
Uncomputable functions

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So there must be some function that is not computable by any program!
Uncomputable functions

We have seen that …

The set of all (Java) programs is countable. The set of all functions $f: \mathbb{N} \rightarrow \{0, 1\}$ is uncountable. So there must be some function that is not computable by any program!

We’ll study one such important function today.
Undecidability of the halting problem

Important problems computers can’t solve.
First, some notation …

We’ll be talking about (Java) code.

\( \text{code}(\mathcal{P}) \) will denote “the code of the program \( \mathcal{P} \)”. 
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Consider this program:

```java
public static boolean P(String x) {
    return x.matches("public .*");
}
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Consider this program:

```java
public static boolean P(String x) {
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```

What is \( P(\text{code}(P)) \)?

true
And now, the halting problem!

The Halting Problem
Given \texttt{code(P)} for any program \( P \) and an input \( x \), output \texttt{true} if \( P \) halts on the input \( x \), and output \texttt{false} if \( P \) does not halt (diverges) on the input \( x \).
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Can’t we determine this by just running \( P \) on \( x \)?
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The Halting Problem

Given code (P) for any program P and an input x, output true if P halts on the input x, and output false if P does not halt (diverges) on the input x.

Can’t we determine this by just running P on x?

No! We can’t tell if P diverged on x or is taking a long time to return.
The halting problem is undecidable

The Halting Problem
Given code (P) for any program P and an input x, output true if P halts on the input x, and output false if P does not halt (diverges) on the input x.

Theorem (due to Alan Turing)
There is no program that solves the halting problem.
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Theorem (due to Alan Turing)
There is no program that solves the halting problem.

In other words, there is no program H(code(P), x) that computes the function described by the halting problem. This function is therefore uncomputable. Because the function outputs a boolean (a yes/no decision), we say that the underlying problem is undecidable.
Proof by contradiction

Suppose that $H$ is a program that solves the halting problem.
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Suppose that $H$ is a program that solves the halting problem.
Then, we can write the program $D$ as follows:

```java
public static void D(String x) {
    if (H(x, x) == true) {
        while (true); // diverge
    } else {
        return; // halt
    }
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Does $D(code(D))$ halt?
Proof by contradiction

Suppose that $\mathcal{H}$ is a program that solves the halting problem. Then, we can write the program $D$ as follows:

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class D {
    public static void D(String x) {
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        }
    }
}
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Does $D(code(D))$ halt?

$\mathcal{H}$ solves the halting problem means the following:

If $D(x)$ halts then $\mathcal{H}(code(D), x)$ is true otherwise $\mathcal{H}(code(D), x)$ is false.
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If $D(x)$ halts then $H(code(D), x)$ is true otherwise $H(code(D), x)$ is false.

Suppose that $D(code(D))$ halts.
Then, by definition of $H$, it must be that $H(code(D), code(D))$ is true. But in that case, $D(code(D))$ doesn’t halt by definition of $D$. 

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Then, by definition of $H$, it must be that $H(code(D),code(D))$ is false.
But in that case, $D(code(D))$ halts by definition of $D$.

So we reach a contradiction in either case.
Therefore, our assumption that $H$ exists must be false. $\square$
Where did the idea for creating $D$ come from?

```java
public static void D(String x) {
    if (H(x, x) == true) {
        while (true); // diverge
    } else {
        return; // halt
    }
}
```

Note that $D$ halts on $\text{code}(P)$

iff $H(\text{code}(P), \text{code}(P))$ outputs false, i.e.,

iff $P$ doesn’t halt on the input $\text{code}(P)$.

Therefore, $D$ differs from every program $P$ on the input $\text{code}(P)$. 
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This sounds like diagonalization!

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public static void D(String x) {
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    }
}
```
“D” is for diagonalization

List all Java programs.
  This list exists because the set of all Java programs is countable.

\[
\begin{array}{c}
  \vdots \\
  P_0 \\
  P_1 \\
  P_2 \\
  P_3 \\
  P_4 \\
\end{array}
\]
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Let \( \langle P \rangle \) stand for \( \text{code}(P) \).
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Let \( \langle P \rangle \) stand for \( \text{code}(P) \).
\((P, x)\) entry is 1 if the program \( P \) halts on input \( x \) and 0 otherwise.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \langle P_0 \rangle )</th>
<th>( \langle P_1 \rangle )</th>
<th>( \langle P_2 \rangle )</th>
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<th>( \langle P_4 \rangle )</th>
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<tr>
<td>( P_0 )</td>
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   This list exists because the set of all Java programs is countable.

Let $\langle P \rangle$ stand for code $(P)$.
   $(P, x)$ entry is 1 if the program $P$ halts on input $x$ and 0 otherwise.

$D$ behaves like the flipped diagonal
   $D(\langle P \rangle) = \neg P(\langle P \rangle)$, and differs from every $P$ in the list.

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   (P, x) entry is 1 if the program P halts on input x and 0 otherwise.

\(D\) behaves like the flipped diagonal
   \(D(⟨P⟩) = \neg P(⟨P⟩)\), and differs from every P in the list.

But the list is complete.
   So if \(D\) isn’t included, it cannot exist!
The halting problem isn’t the only hard problem

To show that a problem \( B \) is undecidable:

Prove that if there were a program deciding \( B \) then there would be a way to build a program deciding the halting problem.
The halting problem isn’t the only hard problem

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That is, prove “$B$ is decidable $\rightarrow$ halting problem is decidable”.
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- By contrapositive, “halting problem is undecidable $\rightarrow$ $B$ is undecidable”.
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Every non-trivial question about program behavior is undecidable.

Termination, equivalence checking, verification, synthesis, …
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Termination, equivalence checking, verification, synthesis, …

But we can often decide these questions in practice!

They are undecidable for arbitrary programs and properties.

Yet decidable for many specific classes of programs and properties.

And when we allow “yes/no/don’t know” answers.
That’s all folks!

Propositional logic.
Boolean logic, circuits, and algebra.
Predicates, quantifiers and predicate logic.
Inference rules and formal proofs for propositional and predicate logic.
English proofs.
Set theory.
Modular arithmetic and prime numbers.
GCD, Euclid’s algorithm, modular inverse, and exponentiation.
Induction and strong induction.
Recursively defined functions and sets.
Structural induction.
Regular expressions.
Context-free grammars and languages.
Relations, composition, and reflexive-transitive closure.
DFAs, NFAs, and product construction for DFAs.
Finite state machines with output.
Minimization algorithm for finite state machines.
Conversion of regular expressions to NFAs.
Subset construction to convert NFAs to DFAs.
Equivalence of DFAs, NFAs, regular expressions.
Method to prove languages are not regular.
Cardinality, countability, and diagonalization.
Undecidability and the halting problem.

Go forth and prove great things!