CSE 311 Lecture 28: Undecidability of the Halting Problem

Emina Torlak and Sami Davies

Topics

Countability and uncomputability

A quick recap of Lecture 27.

Undecidability of the halting problem

Important problems computers can't solve.

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Countable and uncountable sets

Countable set

A set is *countable* iff it has the same cardinality as some subset of $\mathbb{N}$.

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- $\mathbb{N}$ (natural numbers)
- $\mathbb{Z}$ (integers)
- $\mathbb{Q}^+$ (positive rational numbers)
- Σ^* over finite Σ
- All (Java) programs

Shown by dovetailing.

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Uncountable sets

- All real numbers in $[0, 1)$

Shown by diagonalization.

Recall the proof that $[0, 1)$ is uncountable: diagonalization

Suppose for contradiction that there is a list $\{r_0, r_1, r_2, \ldots\}$ of all real numbers in $[0, 1)$.

Consider the digits $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., the n-th digit of r_n for $n \in \mathrm{N}$.

For each such digit x_i, construct the digit $\hat{x}_i$ as follows:

- If $x_i = 1$ then $\hat{x}_i = 0$.
- If $x_i \neq 1$ then $\hat{x}_i = 1$.

Now, consider the number $\hat{r} = 0.\hat{x}_0\hat{x}_1\hat{x}_2\hat{x}_3\ldots$

Note that $r_n \neq \hat{r}$ for any $n \in \mathrm{N}$ because they differ on the n-th digit.

So the list doesn't include $\hat{r}$, which is a contradiction. Thus the set $[0, 1)$ is uncountable.

$$
\begin{array}{ll}
r_0 & 0.\mathbf{5}00000000000\ldots \\
r_1 & 0.3\mathbf{3}3333333333\ldots \\
r_2 & 0.14\mathbf{2}857142857\ldots \\
r_3 & 0.141\mathbf{5}92653589\ldots \\
r_4 & 0.2000\mathbf{0}0000000\ldots \\
\vdots &
\end{array}
$$

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Suppose for contradiction that there is a list $\{f_1, f_2, f_3, \ldots\}$ of functions from $\mathbf{N}$ to $\{0, 1\}$.

$$\begin{array}{ll} f_0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\ f_1 & 1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\ f_2 & 0\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\ f_3 & 0\,1\,1\,1\,0\,1\,1\,1\,0\ldots \\ f_4 & 1\,1\,0\,0\,0\,1\,1\,0\,0\ldots \\ \vdots & \end{array}$$

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Consider the outputs $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., $f_n(n)$ for $n \in \mathbb{N}$.

$$
\begin{array}{ll}
f_0 & \mathbf{0}\;0\;0\;0\;0\;0\;0\;0\;0\ldots \\
f_1 & 1\;\mathbf{1}\;1\;1\;1\;1\;1\;1\;1\ldots \\
f_2 & 0\;1\;\mathbf{0}\;1\;0\;1\;0\;1\;0\ldots \\
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f_4 & 1\;1\;0\;0\;\mathbf{0}\;1\;1\;0\;0\ldots \\
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Now, consider the function $\hat{f}(n) = \hat{x}_n$.

f_0	0 0 0 0 0 0 0 0 0…
f_1	1 1 1 1 1 1 1 1 1…
f_2	0 1 0 1 0 1 0 1 0…
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Note that $f_n \neq \hat{f}$ for any $n \in \mathbb{N}$ because the functions differ on the n-th output.

$$
\begin{array}{ll}
f_0 & \color{magenta}{0}\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
f_1 & 1\,\color{magenta}{1}\,1\,1\,1\,1\,1\,1\,1\ldots \\
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Note that $f_n \neq \hat{f}$ for any $n \in \mathbb{N}$ because the functions differ on the n-th output.

So the list doesn't include $\hat{f}$, which is a contradiction. Thus the set $\{f \mid f: \mathbb{N} \to \{0, 1\}\}$ is uncountable.

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f_2	0 1 0 1 0 1 0 1 0…
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Uncomputable functions

We have seen that …

The set of all (Java) programs is countable.

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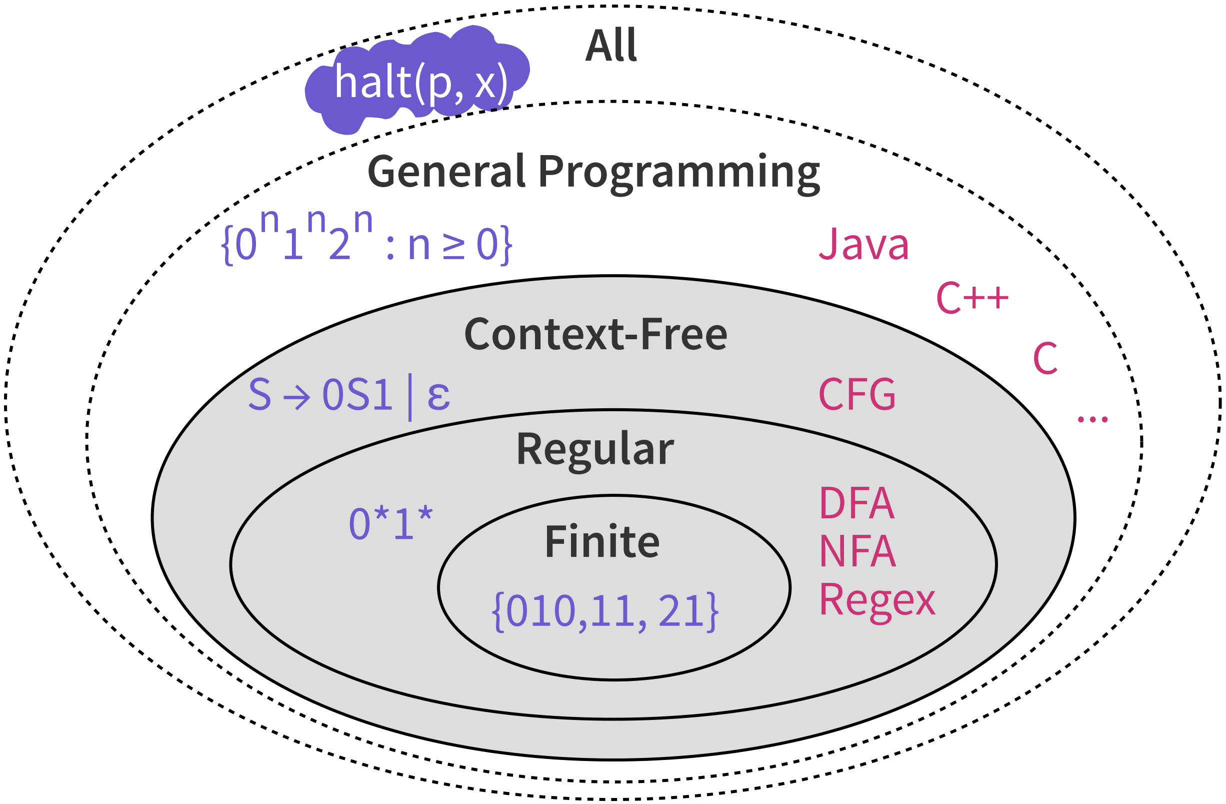
So there must be some function that is not computable by any program!

We'll study one such important function today.



Undecidability of the halting problem

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First, some notation …

We'll be talking about (Java) code.

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public static boolean P(String x) {
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What is `P(code(P))`?

true

And now, the halting problem!

The Halting Problem

Given `code(P)` for any program P and an input x,
output `true` if P halts on the input x, and
output `false` if P does not halt (diverges) on the input x.

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Can't we determine this by just running P on x?

No! We can't tell if P diverged on x or is taking a long time to return.

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There is no program that solves the halting problem.

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There is no program that solves the halting problem.

In other words, there is no program `H(code(P),x)` that computes the function described by the halting problem. This function is therefore uncomputable. Because the function outputs a boolean (a yes/no decision), we say that the underlying problem is *undecidable*.

Proof by contradiction

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Then, we can write the program `D` as follows:

```
public static void D(String x) {
  if (H(x, x) == true) {
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If `D(x)` halts then `H(code(D),x)` is true otherwise `H(code(D),x)` is false.

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Then, by definition of `H`, it must be that `H(code(D),code(D))` is true.

But in that case, `D(code(D))` doesn't halt by definition of `D`.

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But in that case, `D(code(D))` halts by definition of `D`.

So we reach a contradiction in either case.

Therefore, our assumption that `H` exists must be false. $\square$

Where did the idea for creating **D** come from?

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Note that `D` halts on `code(P)`

iff `H(code(P),code(P))` outputs false, i.e.,

iff `P` doesn't halt on the input `code(P)`.

Therefore, `D` differs from every program `P` on the input `code(P)`.

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This sounds like diagonalization!

“D” is for diagonalization

List all Java programs.

This list exists because the set of all Java programs is countable.

P_0
P_1
P_2
P_3
P_4

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	$\langle P_0 \rangle$	$\langle P_1 \rangle$	$\langle P_2 \rangle$	$\langle P_3 \rangle$	$\langle P_4 \rangle$	$\dots$
P_0						
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Let $\langle P \rangle$ stand for `code(P)`.

(P, x) entry is 1 if the program P halts on input x and 0 otherwise.

	$\langle P_0 \rangle$	$\langle P_1 \rangle$	$\langle P_2 \rangle$	$\langle P_3 \rangle$	$\langle P_4 \rangle$	$\dots$
P_0	0	1	1	0	1	$\dots$
P_1	1	1	0	1	0	$\dots$
P_2	1	0	1	0	0	$\dots$
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$D(\langle P \rangle) = \neg P(\langle P \rangle)$, and differs from every P in the list.

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But the list is complete.

So if `D` isn't included, it cannot exist!

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The halting problem isn't the only hard problem

To show that a problem B is undecidable:

Prove that if there were a program deciding B then there would be a way to build a program deciding the halting problem.

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By contrapositive, "halting problem is undecidable $\longrightarrow$ B is undecidable".

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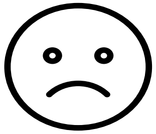
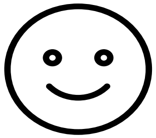


Termination, equivalence checking, verification, synthesis, …

But we can often decide these questions in practice!



They are undecidable for *arbitrary* programs and properties.

Yet decidable for many specific classes of programs and properties.

And when we allow "yes/no/don't know" answers.

That’s all folks!

Propositional logic.
Boolean logic, circuits, and algebra.
Predicates, quantifiers and predicate logic.
Inference rules and formal proofs for propositional and predicate logic.
English proofs.
Set theory.
Modular arithmetic and prime numbers.
GCD, Euclid’s algorithm, modular inverse, and exponentiation.
Induction and strong induction.
Recursively defined functions and sets.
Structural induction.
Regular expressions.
Context-free grammars and languages.
Relations, composition, and reflexive-transitive closure.
DFAs, NFAs, and product construction for DFAs.
Finite state machines with output.
Minimization algorithm for finite state machines.
Conversion of regular expressions to NFAs.
Subset construction to convert NFAs to DFAs.
Equivalence of DFAs, NFAs, regular expressions.
Method to prove languages are not regular.
Cardinality, countability, and diagonalization.
Undecidability and the halting problem.