CSE 311 Lecture 27: Cardinality and Uncomputability

Emina Torlak and Sami Davies
Topics

Course evaluation
   Is open; please tell us what you think!

Proving irregularity
   A quick review of Lecture 26.

Languages and representations
   How powerful are general-purpose programming languages?

Cardinality and countability
   What does it mean for two sets to have the same size?

Uncomputability
   Are there problems computers can’t solve?
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Proving irregularity

A quick review of Lecture 26.
A template for proving that a language $L$ is not regular

① Suppose for contradiction that some DFA $M$ recognizes $L$. 
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② Consider the set $S = \{\ldots\}$. $S$ must be infinite, and for every pair of prefixes $s_a \neq s_b \in S$, there is a suffix $t$ such that one of $s_a t$, $s_b t$ is in $L$ but not the other.
A template for proving that a language $L$ is not regular

1. Suppose for contradiction that some DFA $M$ recognizes $L$.
2. Consider the set $S = \{\ldots\}$.
3. Since $S$ is infinite and $M$ has finitely many states, there must be two strings $s_a, s_b \in S$ such that $s_a \neq s_b$ and both end in the same state of $M$.

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⑤ Since $s_a$ and $s_b$ end in the same state of $M$, then $s_at$ and $s bt$ also end in the same state $q$ of $M$. Since $s_at \in L$ and $s bt \notin L$, $M$ does not recognize $L$.

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5. Since $s_a$ and $s_b$ end in the same state of $M$, then $s_a t$ and $s_b t$ also end in the same state $q$ of $M$. Since $s_a t \in L$ and $s_b t \notin L$, $M$ does not recognize $L$.
6. Since $M$ was arbitrary, no DFA recognizes $L$.

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Consider appending $\epsilon$ to both $0^a$ and $0^b$. 
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Consider appending $1^a$ to both $0^a$ and $0^b$. 
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Languages and representations

How powerful are general-purpose programming languages?
A hierarchy of languages and representations

- Finite
  - Regular
    - Context-Free
      - S → 0S1 | ε
      - 0*1*
  - {010, 11, 21}
A hierarchy of languages and representations

We can think of languages as functions from strings to booleans.
Such a function returns true iff a string is in the language.
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General-purpose programs can represent even more functions.

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General-purpose programs can represent even more functions.
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Are there some functions no program can represent?
That’s what we’ll study in these last two lectures :)

\[
S \rightarrow 0S1 \mid \varepsilon
\]

\[
\{0^n1^n2^n : n \geq 0\}
\]

Finite

\{0,1,2\}

DFA

NFA

Regex

Context-Free

\[
S \rightarrow 0S1 \mid \varepsilon
\]

General Programming

Java

C++

C

...
Cardinality and countability

What does it mean for two sets to have the same size?
Understanding cardinality

What does it mean for two sets to have the same size?
What does it mean for two sets to have the same size? We can establish a one-to-one correspondence between their elements.
Defining one-to-one correspondences

One-to-one (injective) functions
A function $f: A \rightarrow B$ is one-to-one (1-1) if every output corresponds to at most one input:

$$f(x) = f(x') \Rightarrow x = x' \text{ for all } x, x' \in A.$$
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Onto (surjective) functions
A function $f: A \rightarrow B$ is onto if there is at least one input for every output: for each $y \in B$, there is an $x \in A$ such that $f(x) = y$. 

![Diagram of one-to-one functions]

![Diagram of onto functions]
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One-to-one correspondences (bijections)
A function \( f: A \rightarrow B \) is a one-to-one correspondence if it is both one-to-one and onto.
Defining cardinality

Cardinality of two sets

Sets $A$ and $B$ have the same *cardinality* if there is a one-to-one correspondence between them, i.e., there is a bijection $f: A \to B$. 

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Defining cardinality

Cardinality of two sets
Sets $A$ and $B$ have the same cardinality if there is a one-to-one correspondence between them, i.e., there is a bijection $f: A \rightarrow B$.

Example: do $\mathbb{N}$ and even natural numbers have the same cardinality? 
Yes! The 1-1 correspondence is $f(n) = 2n$.

```
0  1  2  3  4  5  6  7  8  9  10  ...
0  2  4  6  8 10 12 14 16 18 20  ...
```
Countable sets

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Equivalently, we can say that ...

A set $S$ is countable iff there is an onto function $g : \mathbb{N} \rightarrow S$. 
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A set \( S \) is countable iff there is an *onto* function \( g : \mathbb{N} \to S \).

And we can also say that …

A set \( S \) is countable iff we can order its elements: \( S = \{x_0, x_1, x_2, \ldots \} \).
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Example: is the set $\mathbb{Z}$ of all integers countable?
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Example: is the set \( \mathbb{Z} \) of all integers countable?

\[
\begin{array}{cccccccccccc}
N & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
Z & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & 5 & -5 & \ldots \\
\end{array}
\]
Is the set $\mathbb{Q}^+$ of positive rationals countable?

There are infinitely many rationals between any two rational numbers.
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\[
\begin{array}{cccccccc}
1/1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & \ldots \\
2/1 & 2/2 & 2/3 & 2/4 & 2/5 & 2/6 & 2/7 & 2/8 & \ldots \\
4/1 & 4/2 & 4/3 & 4/4 & 4/5 & 4/6 & 4/7 & 4/8 & \ldots \\
5/1 & 5/2 & 5/3 & 5/4 & 5/5 & 5/6 & 5/7 & 5/8 & \ldots \\
7/1 & 7/2 & 7/3 & 7/4 & 7/5 & 7/6 & 7/7 & 7/8 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{array}
\]
Is the set $\mathbb{Q}^+$ of positive rationals countable?

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Counting $\mathbb{Q}^+$ with dovetailing

The set of all positive rational numbers is countable.

$$\mathbb{Q}^+ = \{1/1, 2/1, 1/2, 3/1, 2/2, 1/3, 4/1, 2/3, 3/2, \ldots\}$$

List elements in the order of the sum of the numerator and denominator, breaking ties according to the denominator.

Only $k$ pairs of positive numbers add up to $k + 1$, so every positive rational number comes up some point.

This technique is called \textit{dovetailing}.
Σ * is countable for every finite Σ

How would we show this?

- Alphabetical / lexicographic order doesn’t work (infinitely many A’s):
  A, AA, AAA, AAAA, AAAAA, …
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Use dovetailing again!
  List strings in the order of length, breaking ties lexicographically.
  There are only |Σ|^k strings on length k.
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Use dovetailing again!
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For example, \{0, 1\} * is countable:
   \{ε, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, …\}
The set of all Java programs is countable
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Java programs are just strings in $\Sigma^*$ where $\Sigma$ is the alphabet of ASCII characters. Since $\Sigma^*$ is countable, so is the set of all Java programs.
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This is true for other programming languages too: C, C++, Python, JavaScript, etc.
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So, is everything countable?
Real numbers are not countable

Theorem [due to Cantor]

The set of real numbers between 0 and 1, [0, 1), is not countable.
Real numbers are not countable

Theorem [due to Cantor]
The set of real numbers between 0 and 1, \([0, 1)\), is not countable.

Proof will be by contradiction. Using a method called diagonalization.
Proof that \([0, 1)\) is uncountable: preliminaries

First, note that every number in \([0, 1)\) has an infinite decimal expansion:

\[
\begin{align*}
1/2 &= 0.50000000000000000000000\ldots \\
1/3 &= 0.33333333333333333333333\ldots \\
1/7 &= 0.14285714285714285714285\ldots \\
\pi - 3 &= 0.14159265358979323846264\ldots \\
1/5 &= 0.19999999999999999999999\ldots \\
&= 0.20000000000000000000000\ldots
\end{align*}
\]

This representation is unique except for the cases where the decimal expansion ends in all 0’s or all 9’s. We will use the all 0’s representation.
Proof that \([0, 1)\) is uncountable: diagonalization

Suppose for contradiction that there is a list \(\{r_0, r_1, r_2, \ldots\}\) of all real numbers in \([0, 1)\).

\[
\begin{align*}
  r_0 &\quad 0.500000000000000\ldots \\
  r_1 &\quad 0.333333333333333\ldots \\
  r_2 &\quad 0.142857142857142\ldots \\
  r_3 &\quad 0.141592653589793\ldots \\
  r_4 &\quad 0.200000000000000\ldots \\
  \vdots
\end{align*}
\]
Proof that \([0, 1)\) is uncountable: diagonalization

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Consider the digits \(x_0, x_1, x_2, x_3, \ldots\) on the diagonal of this list, i.e., the \(n\)-th digit of \(r_n\) for \(n \in \mathbb{N}\).

\[
\begin{align*}
  r_0 & \quad 0.500000000000000\ldots \\
  r_1 & \quad 0.333333333333333\ldots \\
  r_2 & \quad 0.142857142857\ldots \\
  r_3 & \quad 0.141592653589\ldots \\
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For each such digit \(x_i\), construct the digit \(\hat{x}_i\) as follows:

- If \(x_i = 1\) then \(\hat{x}_i = 0\).
- If \(x_i \neq 1\) then \(\hat{x}_i = 1\).

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Now, consider the number \(\hat{r} = 0.\hat{x}_0\hat{x}_1\hat{x}_2\hat{x}_3\ldots\)

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Note that \(r_n \neq \hat{r}\) for any \(n \in \mathbb{N}\) because they differ on the \(n\)-th digit.
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So the list doesn’t include \(\hat{r}\), which is a contradiction. Thus the set \([0, 1)\) is uncountable.
The set of all functions \( f: \mathbb{N} \to \{0, 1\} \) is uncountable.
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Suppose for contradiction that there is a list $\{f_1, f_2, f_3, \ldots\}$ of functions from $\mathbb{N}$ to $\{0, 1\}$.

\[
\begin{align*}
  f_0 & \quad 000000000000\ldots \\
  f_1 & \quad 11111111111\ldots \\
  f_2 & \quad 01010101010\ldots \\
  f_3 & \quad 0111011110\ldots \\
  f_4 & \quad 1100011000\ldots \\
  \vdots
\end{align*}
\]
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Consider the outputs $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., $f_n(n)$ for $n \in \mathbb{N}$.

\[
\begin{align*}
   f_0 &= 0000000000... \\
   f_1 &= 1111111111... \\
   f_2 &= 0101010101... \\
   f_3 &= 0111011110... \\
   f_4 &= 1100011000... \\
   \vdots
\end{align*}
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Consider the outputs $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., $f_n(n)$ for $n \in \mathbb{N}$.

For each such output $x_i$, construct $\hat{x}_i$ as follows:

- If $x_i = 1$ then $\hat{x}_i = 0$.
- If $x_i \neq 1$ then $\hat{x}_i = 1$.

\[
\begin{align*}
  f_0 & \quad 0000000000\ldots \\
  f_1 & \quad 1111111111\ldots \\
  f_2 & \quad 0101010101\ldots \\
  f_3 & \quad 0111011110\ldots \\
  f_4 & \quad 110001100\ldots \\
  \vdots & 
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Now, consider the function $\hat{f}(n) = \hat{x}_n$. 

| \(f_0\) | 0000000000... |
|\(f_1\) | 1111111111... |
|\(f_2\) | 01010101010... |
|\(f_3\) | 011101110110... |
|\(f_4\) | 110001100110... |
|... | ... |
The set of all functions $f: \mathbb{N} \rightarrow \{0, 1\}$ is uncountable

Suppose for contradiction that there is a list $\{f_1, f_2, f_3, \ldots\}$ of functions from $\mathbb{N}$ to $\{0, 1\}$.

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Now, consider the function $\hat{f}(n) = \hat{x}_n$.

Note that $f_n \neq \hat{f}$ for any $n \in \mathbb{N}$ because the functions differ on the $n$-th output.
The set of all functions $f: \mathbb{N} \rightarrow \{0, 1\}$ is uncountable

Suppose for contradiction that there is a list \{\(f_1, f_2, f_3, \ldots\)\} of functions from \(\mathbb{N}\) to \(\{0, 1\}\).

Consider the outputs \(x_0, x_1, x_2, x_3, \ldots\) on the diagonal of this list, i.e., \(f_n(n)\) for \(n \in \mathbb{N}\).

For each such output \(x_i\), construct \(\hat{x}_i\) as follows:

- If \(x_i = 1\) then \(\hat{x}_i = 0\).
- If \(x_i \neq 1\) then \(\hat{x}_i = 1\).

Now, consider the function \(\hat{f}(n) = \hat{x}_n\).

Note that \(f_n \neq \hat{f}\) for any \(n \in \mathbb{N}\) because the functions differ on the \(n\)-th output.

So the list doesn’t include \(\hat{f}\), which is a contradiction. Thus the set \(\{f \mid f: \mathbb{N} \rightarrow \{0, 1\}\}\) is uncountable.
Uncomputability

Are there problems computers can’t solve?
Uncomputable functions

We have seen that …

The set of all (Java) programs is countable.
The set of all functions \( f : \mathbb{N} \rightarrow \{0, 1\} \) is uncountable.
We have seen that …
  The set of all (Java) programs is countable.
  The set of all functions $f: \mathbb{N} \rightarrow \{0, 1\}$ is uncountable.

So there must be some function $f: \mathbb{N} \rightarrow \{0, 1\}$ that is not computable by any program! We’ll study one such function next time.
Summary

Cardinality and countability.
Two sets have the same cardinality if there is a bijection between them. A set is countable iff it has the same cardinality as some subset of \( \mathbb{N} \). Use dovetailing to show that a set is countable and diagonalization to show that it’s uncountable.

Computability.
Countability of programs and uncountability of functions \( f: \mathbb{N} \rightarrow \{0, 1\} \) tells us that there is some function that can’t be computed by any program!