CSE 311 Lecture 26: Limitations of DFAs, NFAs, and Regular Expressions

Emina Torlak and Sami Davies

Topics

DFAs $\equiv$ NFAs $\equiv$ regular expressions

A quick review of Lecture 25.

Languages and representations

Regular, context-free, and other languages.

Proving irregularity

A proof template for showing that a language is not regular.

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Equivalence of DFAs, NFAs, and regular expressions

We have shown how to build an optimal DFA for every regular expression.

Build an NFA.

Convert the NFA to a DFA using the subset construction.

Minimize the resulting DFA.

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Theorem

A language is recognized by a DFA (or NFA) if and only if it has a regular expression.

You need to know this fact but we won't ask you anything about the "only if" direction from DFAs/NFAs to regular expressions.

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Languages represented by NFAs, DFAs, and regular expressions are called *regular languages*.

Languages and representations

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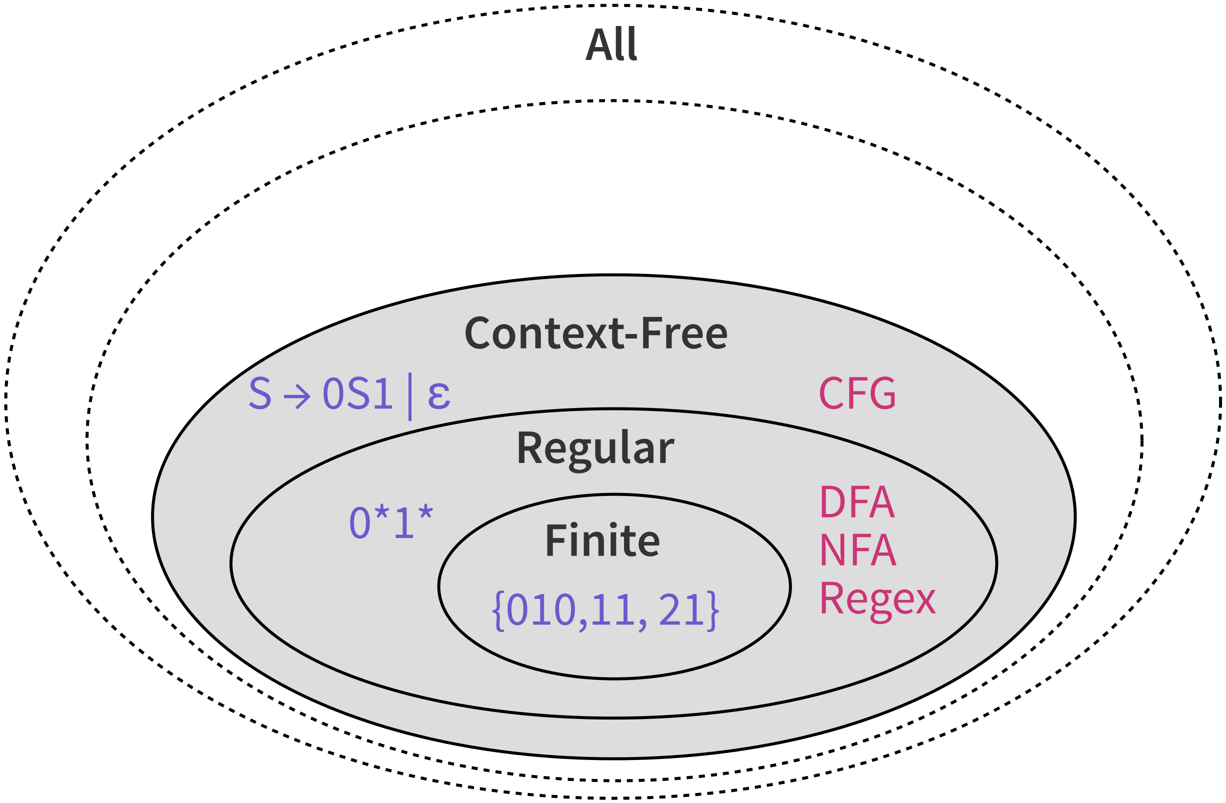
A hierarchy of languages and representations



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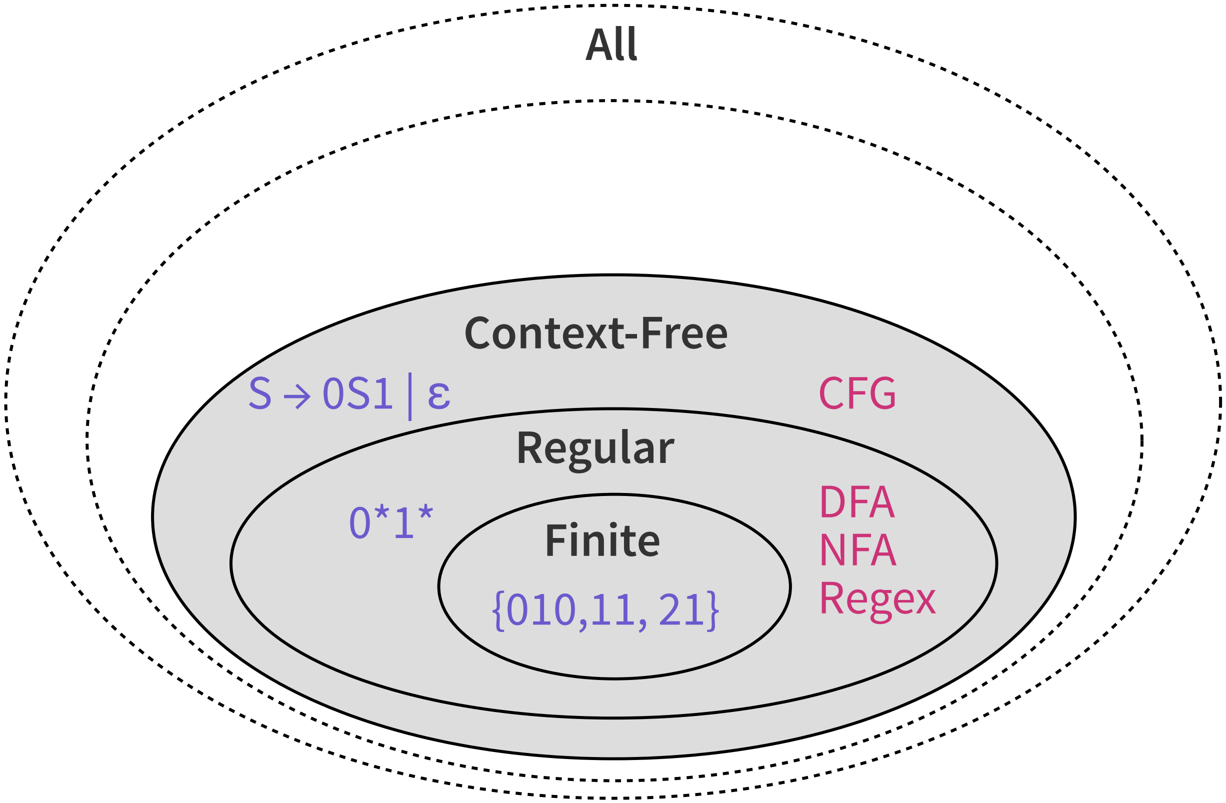
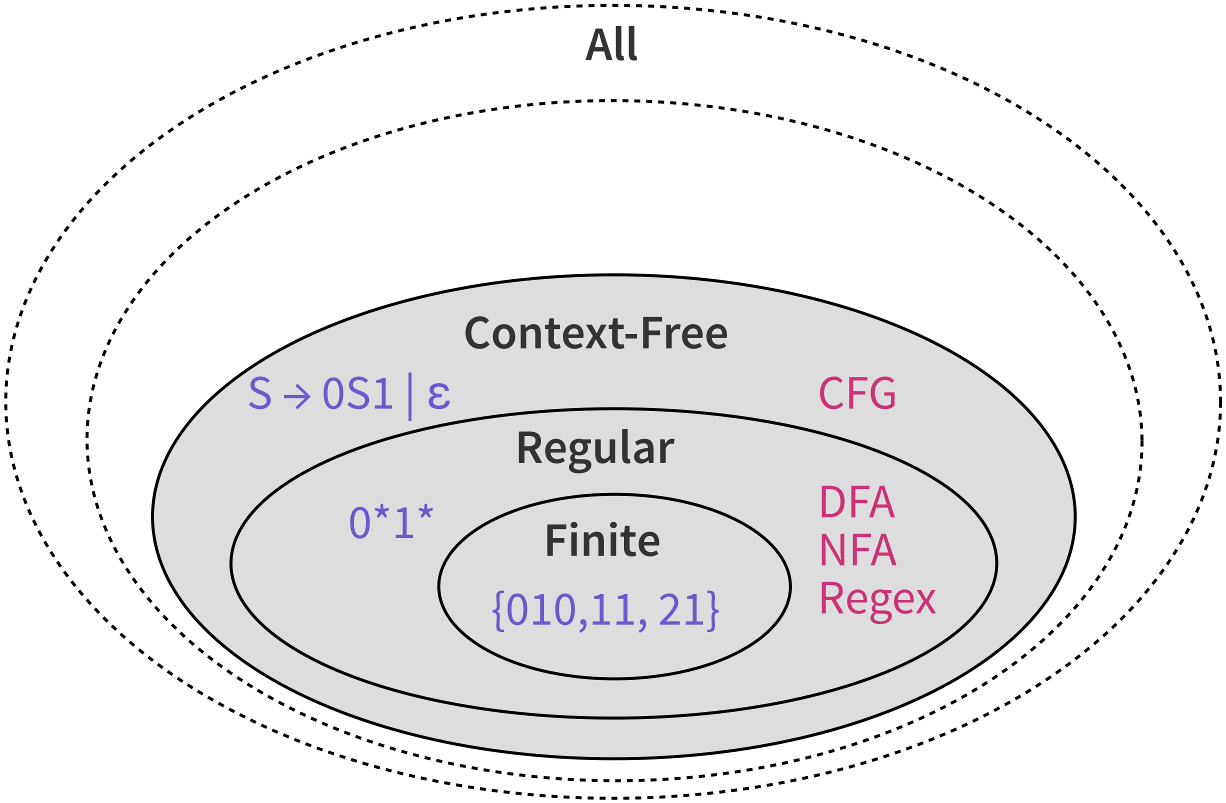


How do we prove that all finite languages are regular?

A hierarchy of languages and representations



How do we prove that all finite languages are regular? By showing how to construct a DFA/NFA/RegEx for every finite language!

Converting a finite language to a regular expression

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Convert each string in the language L to a regular expression.

This is just each string itself.

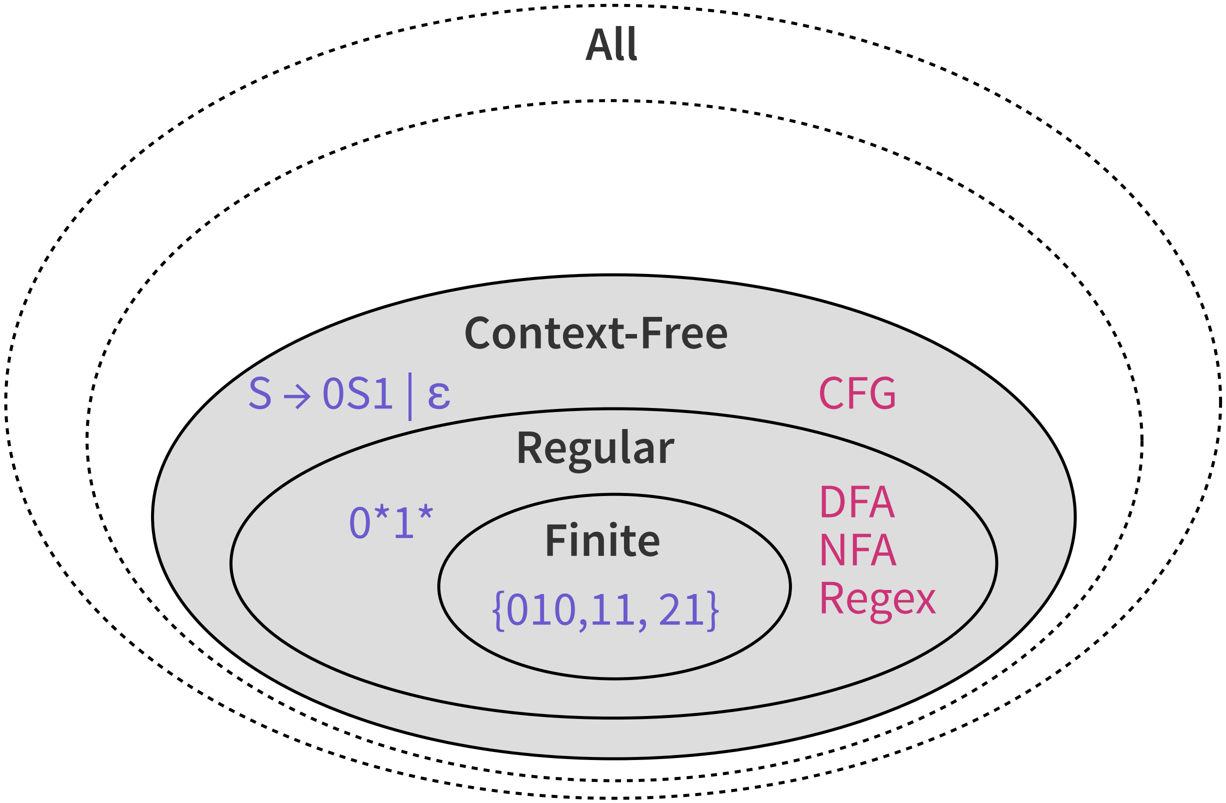
Then put these regular expressions together using the $\cup$ operator.

The resulting regular expression accepts exactly the strings in L.

Example

$$\{010, 11, 21\} \longrightarrow 010 \cup 11 \cup 21$$

Back to languages and representations …



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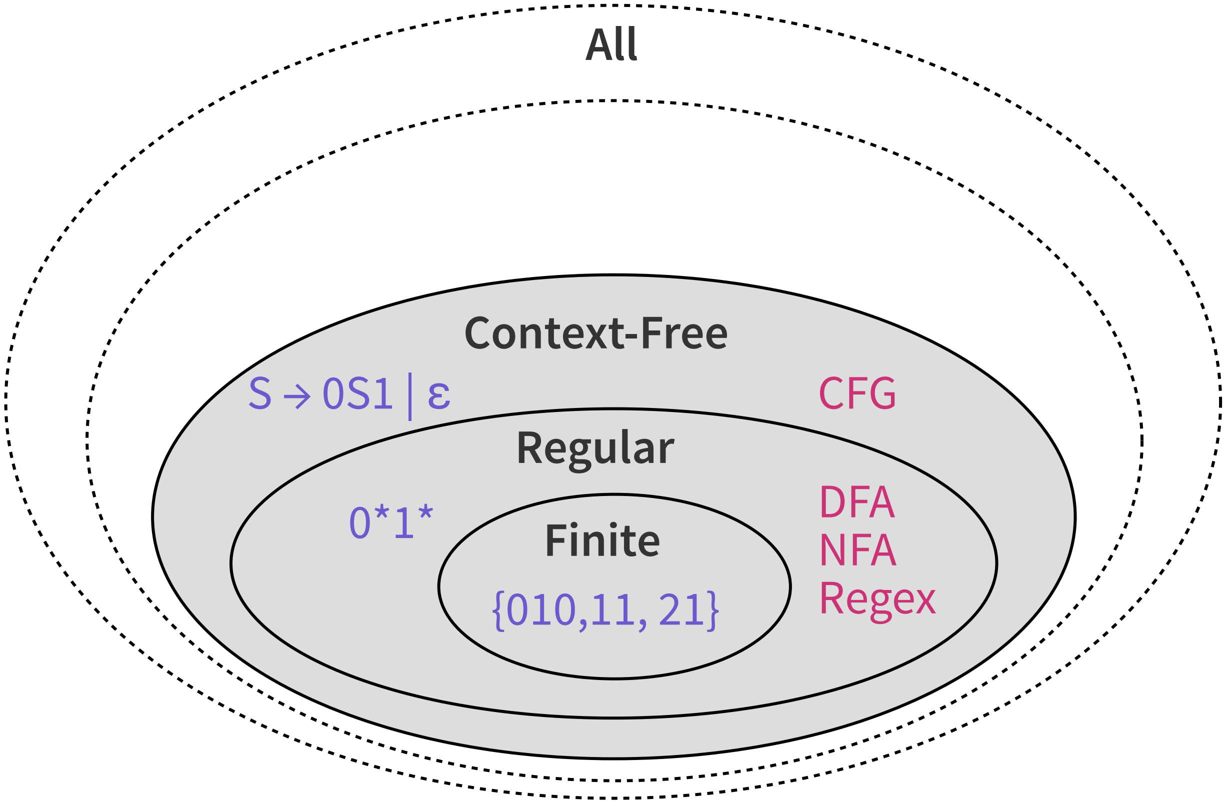

How do we prove that all regular languages are context-free?

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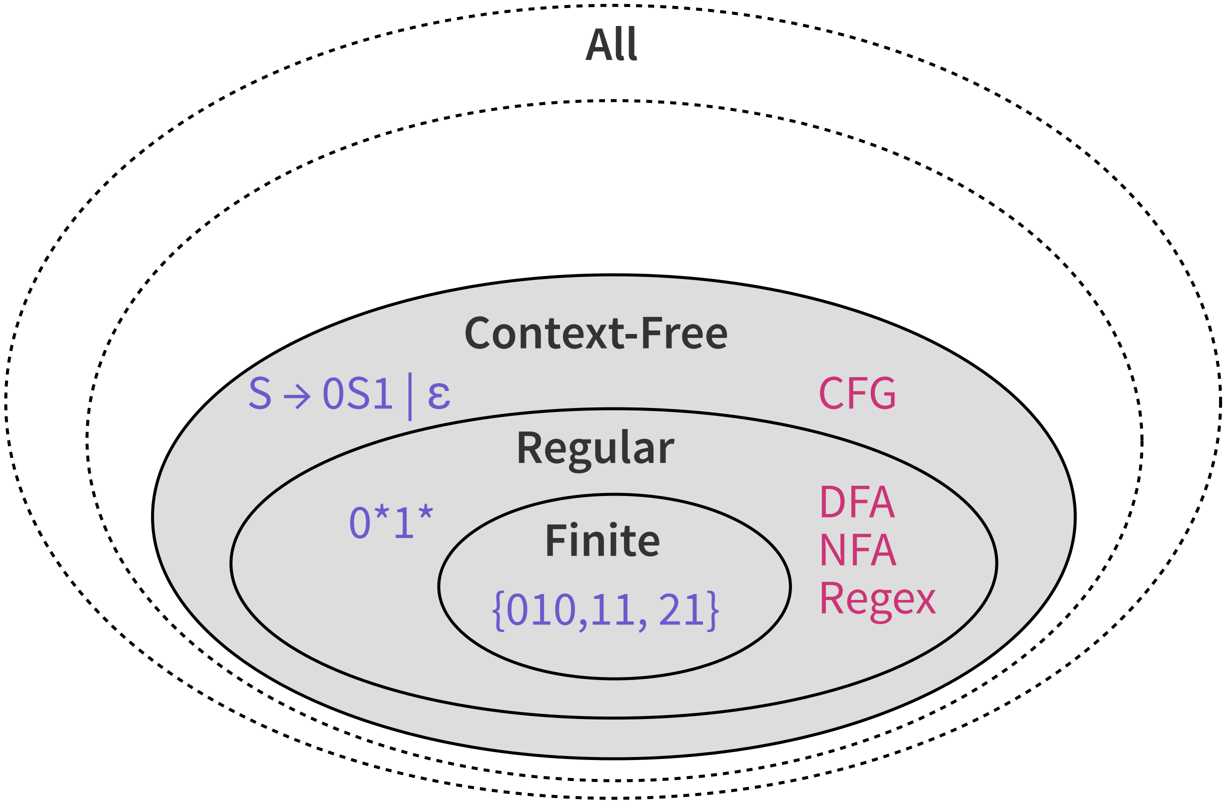


How do we prove that all regular languages are context-free? By showing how to construct a CFG for every regular language!

Converting a regular expression to a CFG

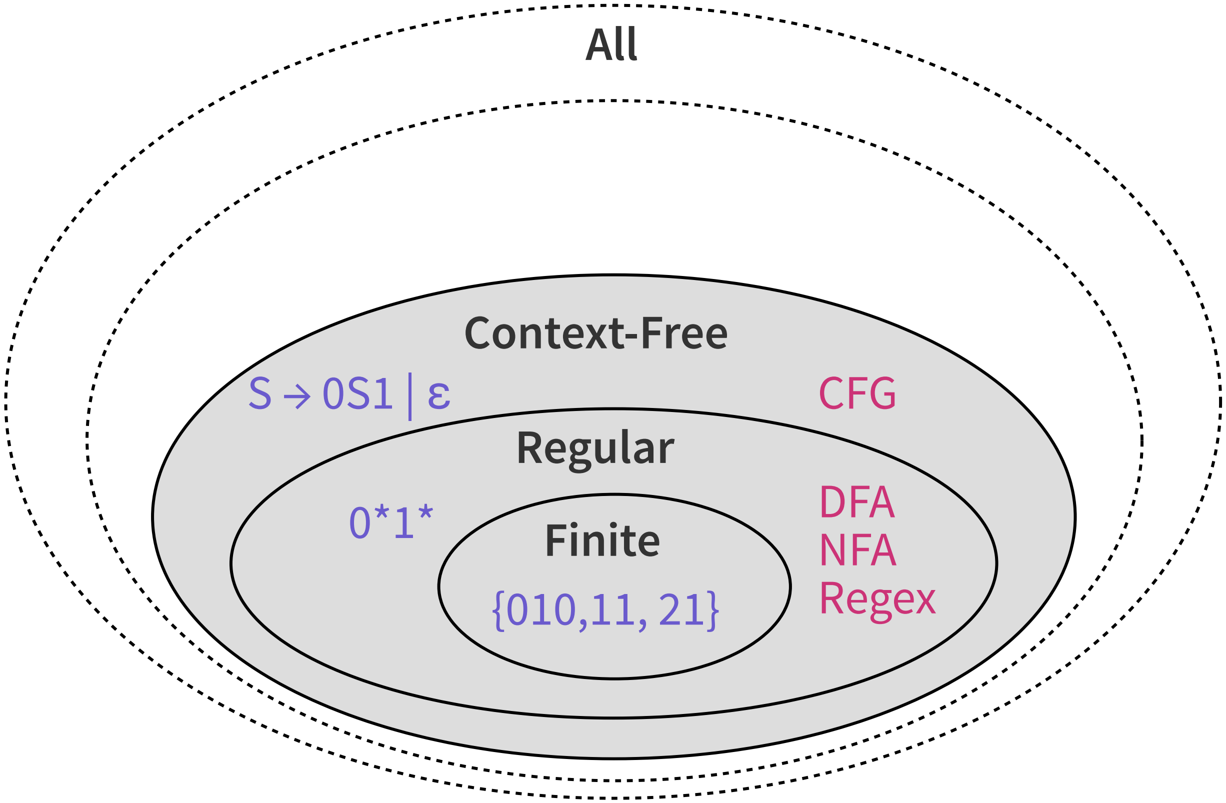
Use the following function on regular expressions:

$\mathsf{cfg}(\emptyset) =$ the CFG with no productions.
$\mathsf{cfg}(\varepsilon) =$ the CFG with just the production $S_\varepsilon \to \varepsilon$.
$\mathsf{cfg}(a) =$ the CFG with just the production $S_a \to a$ for every $a \in \Sigma$.
$\mathsf{cfg}(AB) =$ the CFG with the productions $\mathsf{cfg}(A)$, $\mathsf{cfg}(B)$, and $S_{AB} \to S_A S_B$.
$\mathsf{cfg}(A \cup B) =$ the CFG with the productions $\mathsf{cfg}(A)$, $\mathsf{cfg}(B)$, and $S_{A\cup B} \to S_A \mid S_B$.
$\mathsf{cfg}(A^*) =$ the CFG with the productions $\mathsf{cfg}(A)$ and $S_{A*} \to \varepsilon \mid S_A S_{A*}$.

Example: $\mathsf{cfg}((1 \cup 0)^*0)$

$$\begin{aligned}
S_{(1\cup 0)^*0} &\to S_{(1\cup 0)^*} S_0 \\
S_{(1\cup 0)^*} &\to \varepsilon \mid S_{1\cup 0} S_{(1\cup 0)^*} \\
S_{(1\cup 0)} &\to S_1 \mid S_0 \\
S_1 &\to 1 \\
S_0 &\to 0
\end{aligned}$$

Back again to languages and representations …



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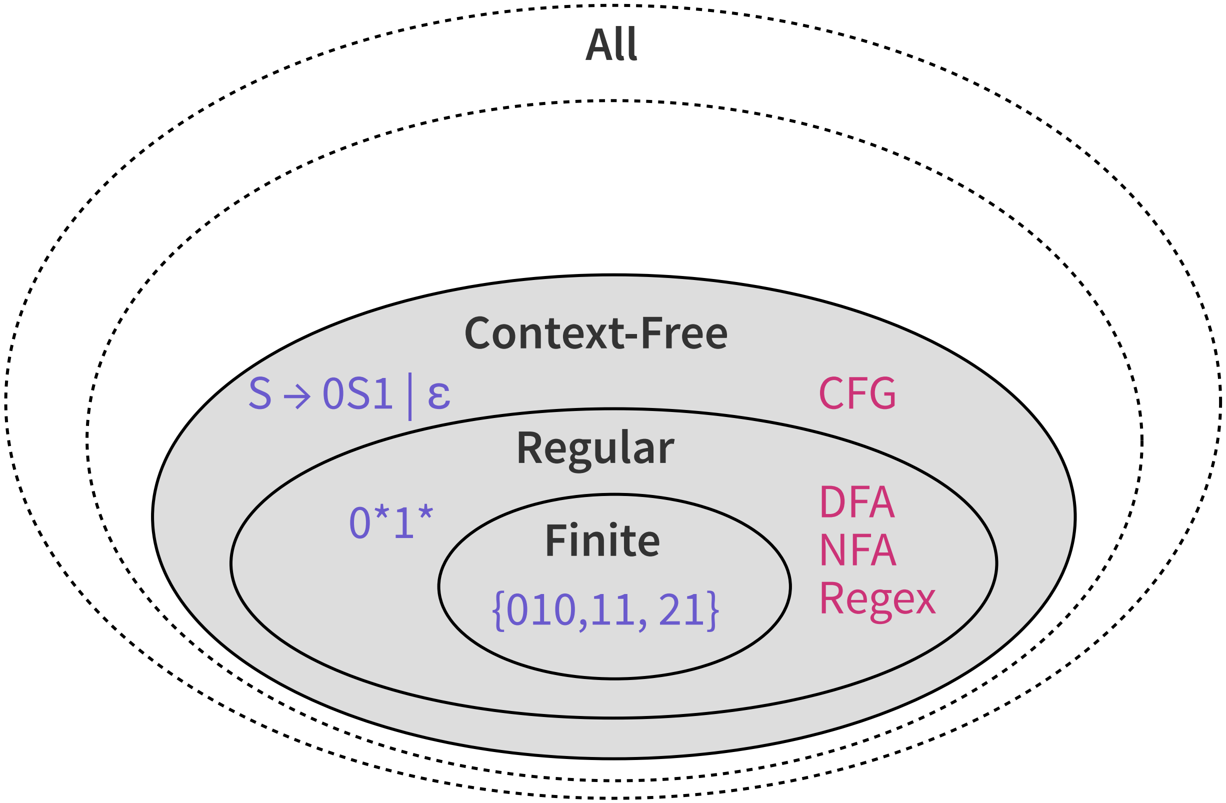

We saw in Lecture 21 that the language B of all binary palindromes can be represented by a CFG. We also said that B can't be represented by any regular expression. How would you prove that?

Why isn't B (binary palindromes) a regular language?

If B were regular, we could express it as a DFA/NFA/RegEx.

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Now, recall that B consists of infinitely many strings $wv\overline{w}$

where $\overline{w}$ is the reverse of w and $v \in \{\varepsilon, \text{“}0\text{”}, \text{“}1\text{”}\}$.

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But there are infinitely many possible w's and finitely many DFA states!

This is the intuition for why B is not regular. Let's see how to turn this intuition into a formal proof.

A strategy for proving that B is not regular

Proof by contradiction:

Assume that B is regular.

Therefore, there is a DFA M that recognizes B.

Show that M accepts or rejects a string it shouldn't.

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Key Idea 1

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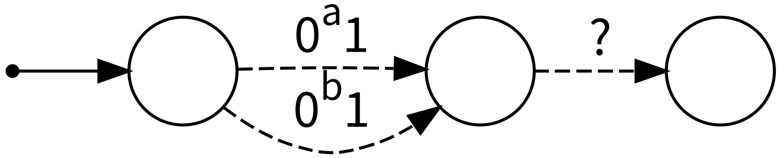

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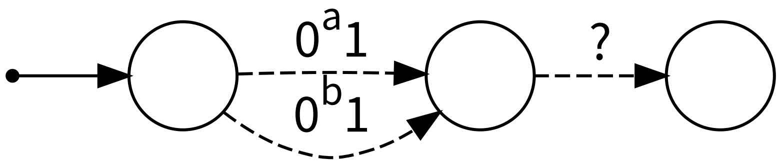

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The machine M has finitely many states, and since the strings in B have infinitely many distinct prefixes, two of them must collide!

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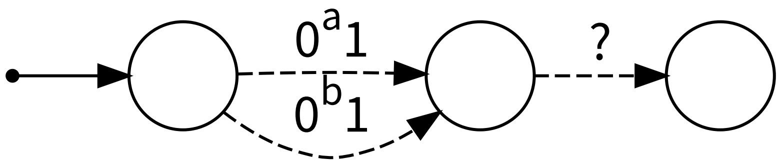

Key Idea 2

The machine M has finitely many states, and since the strings in B have infinitely many distinct prefixes, two of them must collide!

We choose an infinite set S of prefixes. This choice must ensure that for *every pair* of prefixes in $s_a \neq s_b \in S$, there is a suffix t such that one of of $s_a t$, $s_b t$ is in B but not the other.

$$S = \{1, 01, 001, 0001, \ldots\}$$
$$= \{0^n 1 : n \geq 0\}$$

Proving that B is not regular

Suppose that some DFA M recognizes B. We show M accepts or rejects a string it shouldn't. Consider the set $S = \{0^n 1 : n \geq 0\}$.

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Important: We don't get to choose a and b! We just know they exist.

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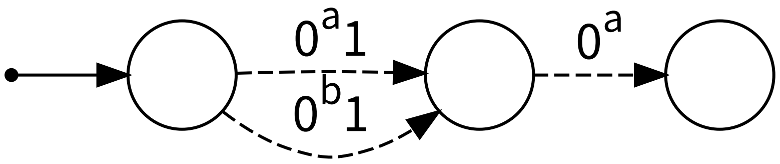
Now, consider appending 0^a to both strings.



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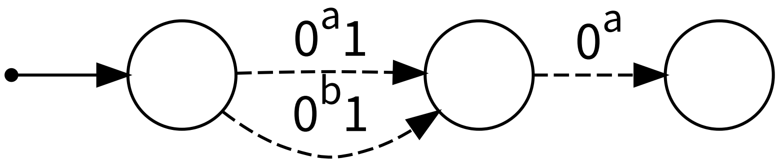

Since $0^a 1$ and $0^b 1$ end in the same state, so do $0^a 10^a$ and $0^b 10^a$, call it q. But then M must make a mistake: q needs to be an accept state since $0^a 10^a \in B$, but then M would accept $0^b 10^a \notin B$, which is an error. $\square$

Proving irregularity

A proof template for showing that a language is not regular.

A template for proving that a language L is not regular

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S must be infinite, and for *every pair* of prefixes $s_a \neq s_b \in S$, there is a suffix t such that one of of $s_a t, s_b t$ is in L but not the other.

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A fun fact about this proof method

Suppose that for a language L, the set S is a *largest* set of prefix strings with the property that for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t$, $s_b t$ is in L but the other isn't.

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If S is infinite, then L is not regular.

If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S.

Summary

DFAs $\equiv$ NFAs $\equiv$ regular expressions.

We've shown how to go from a regular expression to an NFA to a DFA.
(And going from an NFA to a DFA can lead to exponential blow-up.)
But we won't show how to go from an NFA/DFA to a regular expression.

Finite $\subset$ regular $\subset$ context-free languages.

To show that a language is regular, construct a DFA/NFA/RegEx for it.
To show that it is not regular, use the proof method from this lecture.