CSE 311 Lecture 24: FSM Minimization and NFAs

Emina Torlak and Sami Davies

Topics

FSM with output example

Review the vending machine example

FSM minimization

Algorithm and examples.

Nondeterministic finite automata (NFAs)

Definition and examples.

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Review the vending machine example

Vending Machine

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Enter 15 cents in dimes or nickels.

Press S (Snickers) or B (Butterfinger) for a candy bar.

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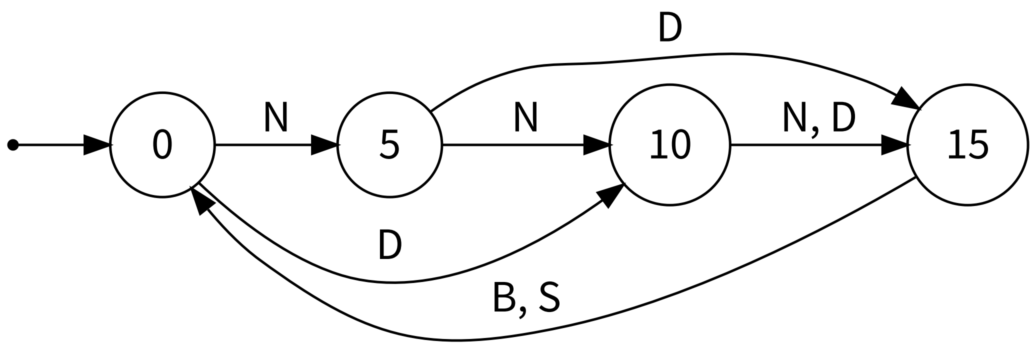
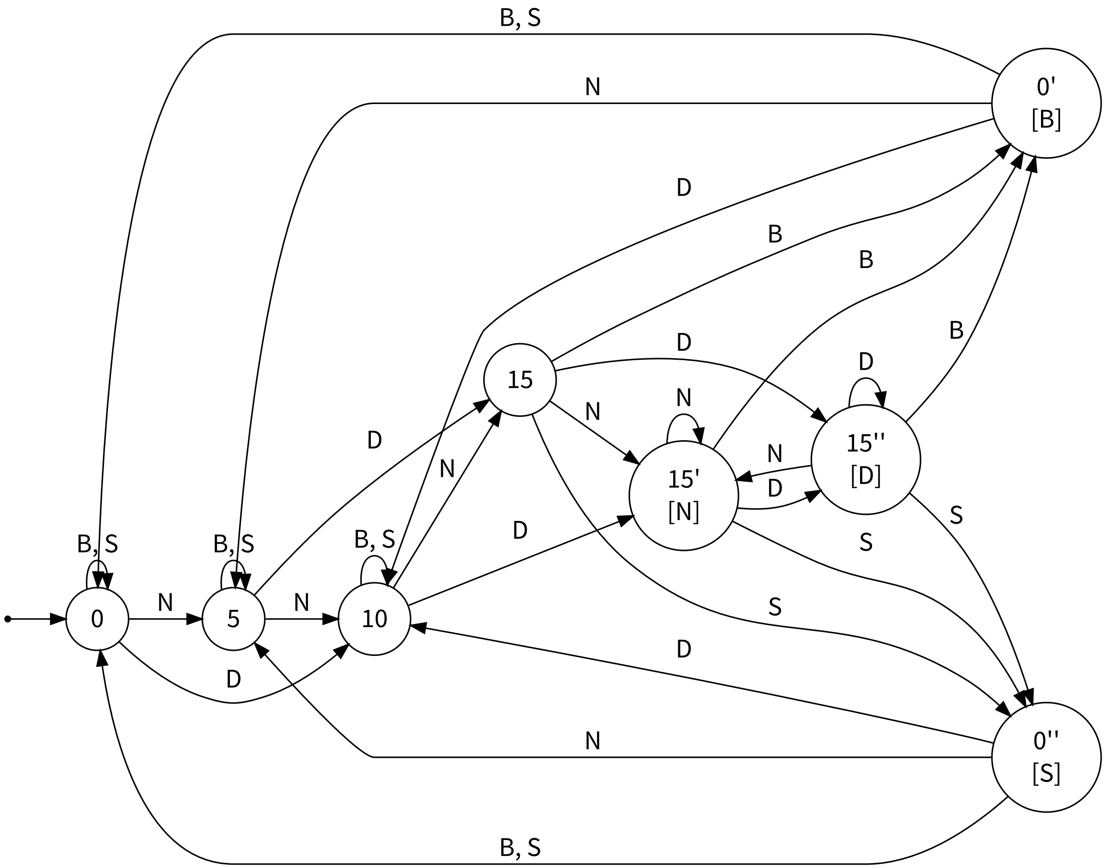


Basic transitions on N (nickel), D (dime), B (Butterfinger), and S (Snickers).

Example: complete vending machine with output



Add transitions to cover all symbols for each state.

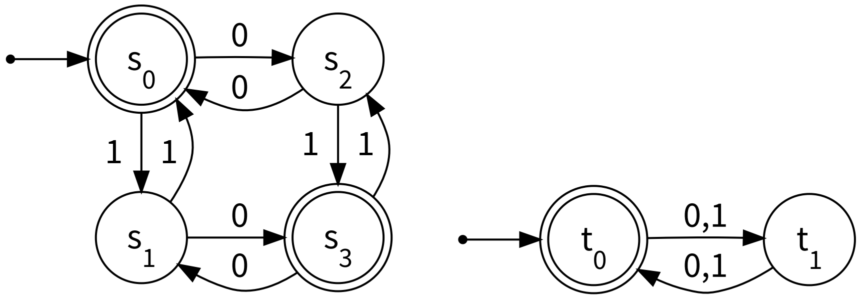
FSM minimization

Algorithm and examples.

How would you compare two FSMs?

Suppose we are given two FSMs over the same input alphabet.

How can we tell if they are *equivalent*, i.e., accept the same language?



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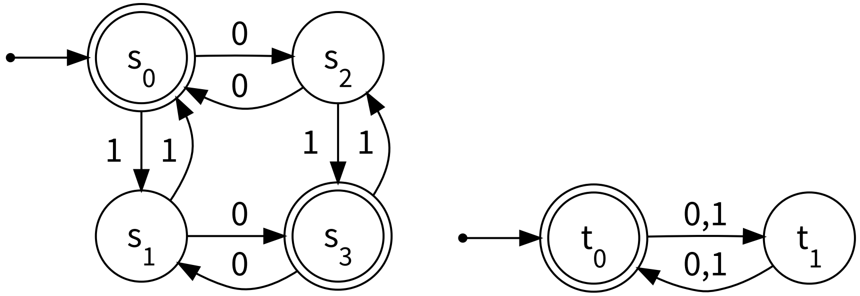

This is exactly the question answered by `grinch`! And many important practical applications beyond autograding, e.g., efficient implementation of `grep` :)

FSM minimization

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So to check if two FSMs are equivalent:

(1) Minimize them, and

(2) Compare the minimal FSMs for equality (modulo renaming of states).

FSM minimization algorithm

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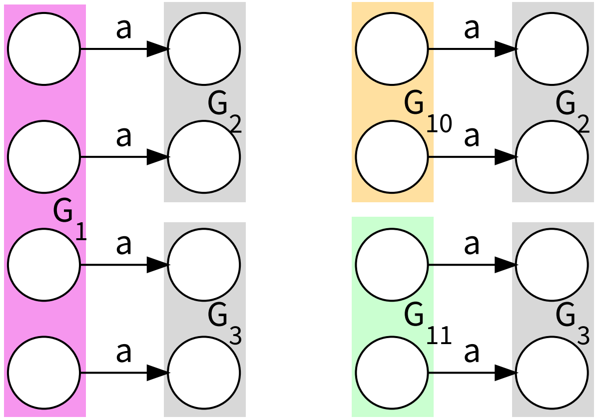

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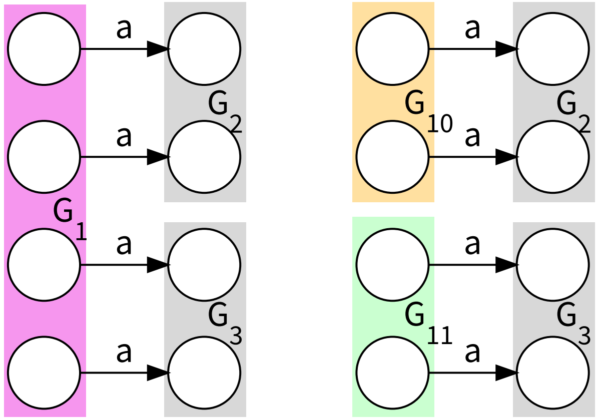
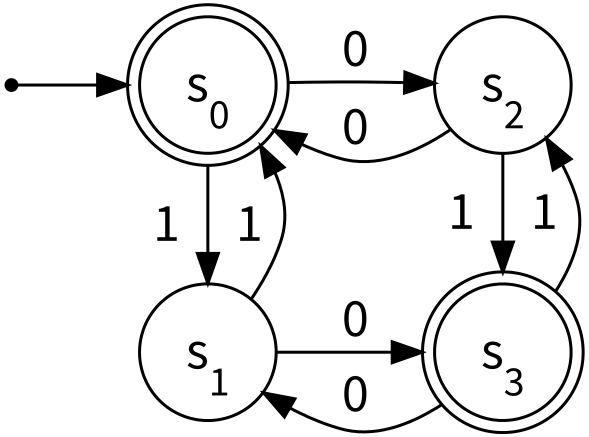


3. Convert groups to states and collapse edges with corresponding labels.

Example: minimizing a DFA



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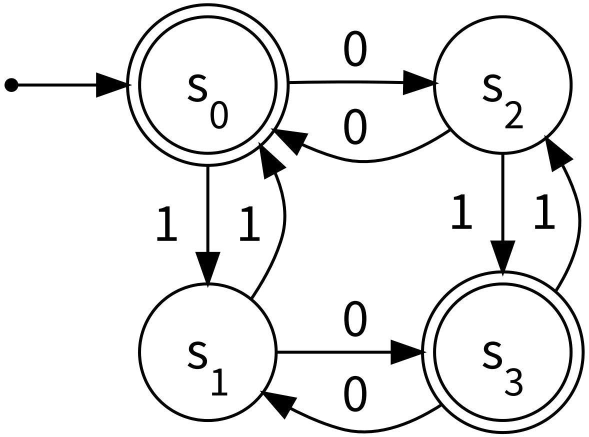

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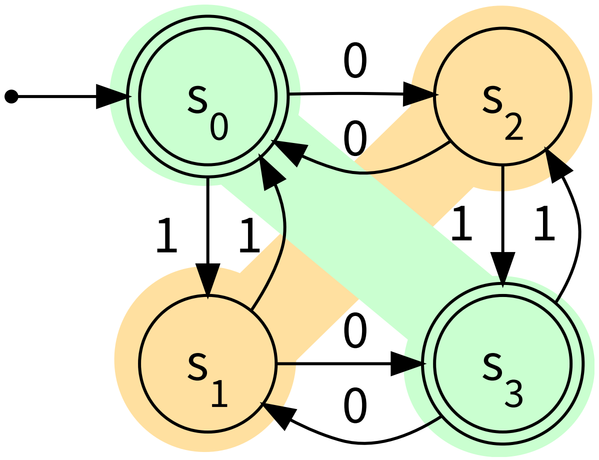

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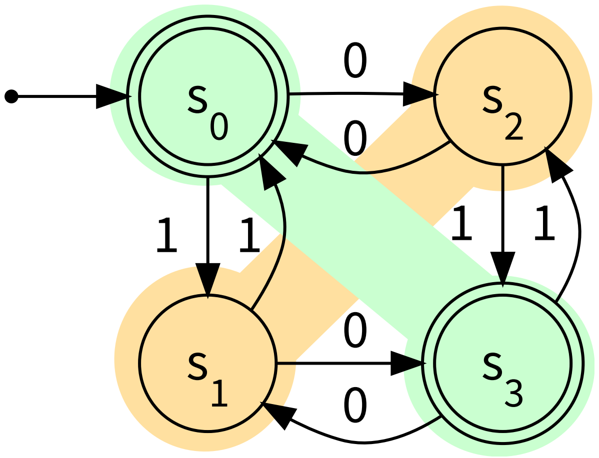

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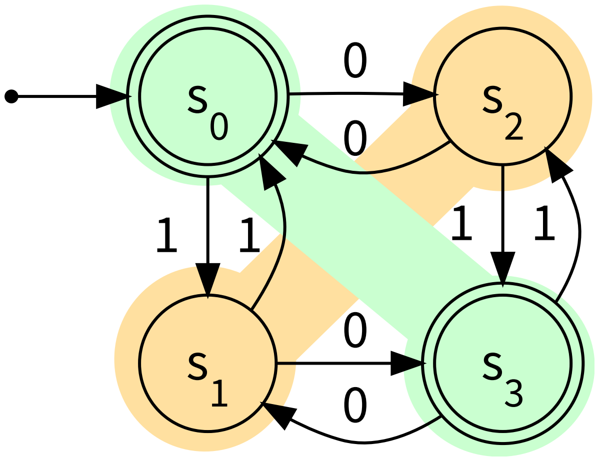

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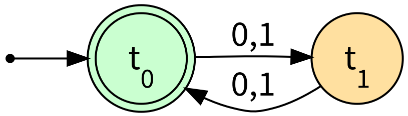
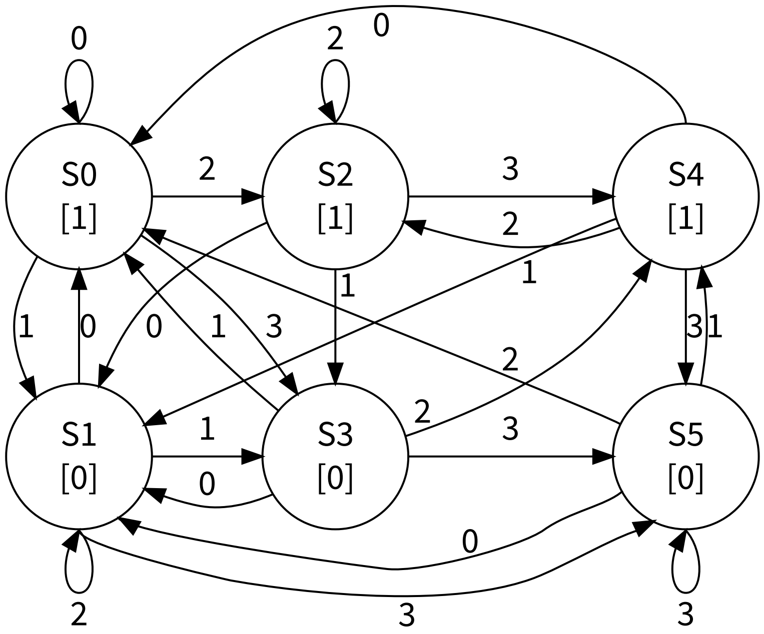


Example: minimizing an FSM with output



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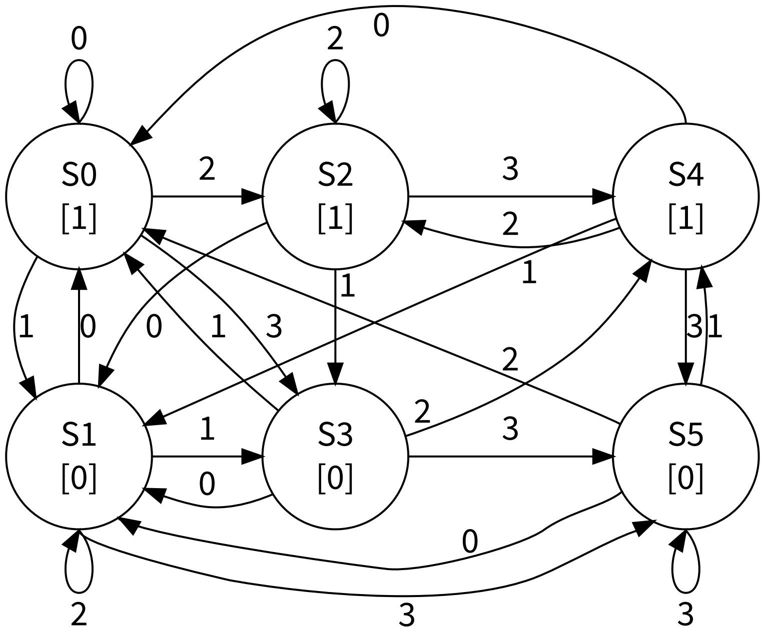

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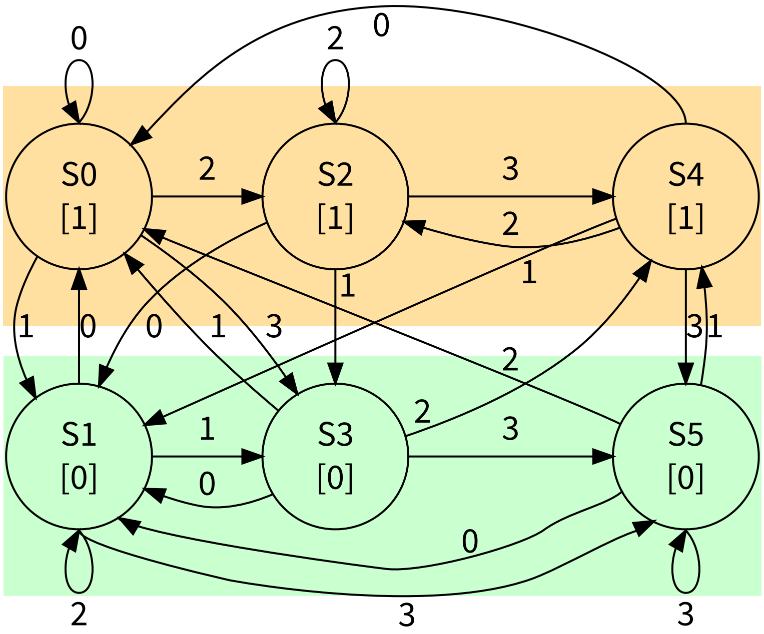

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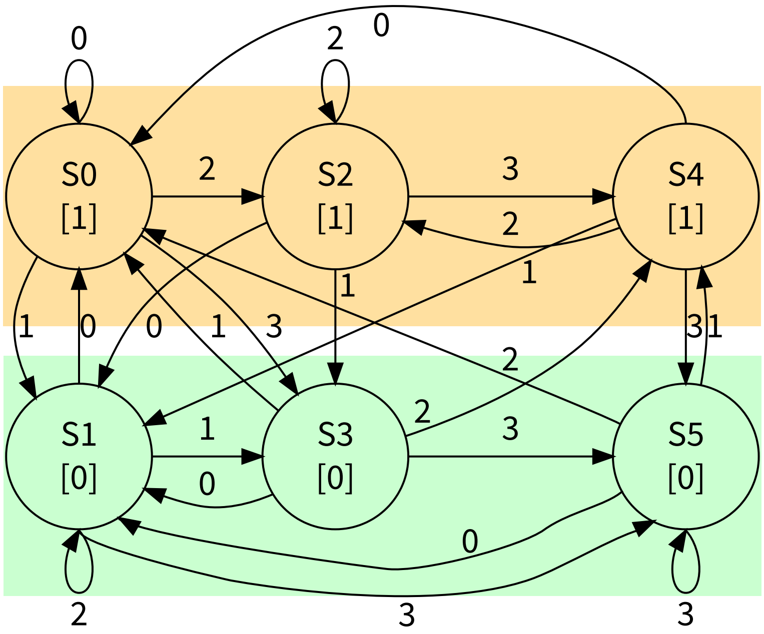

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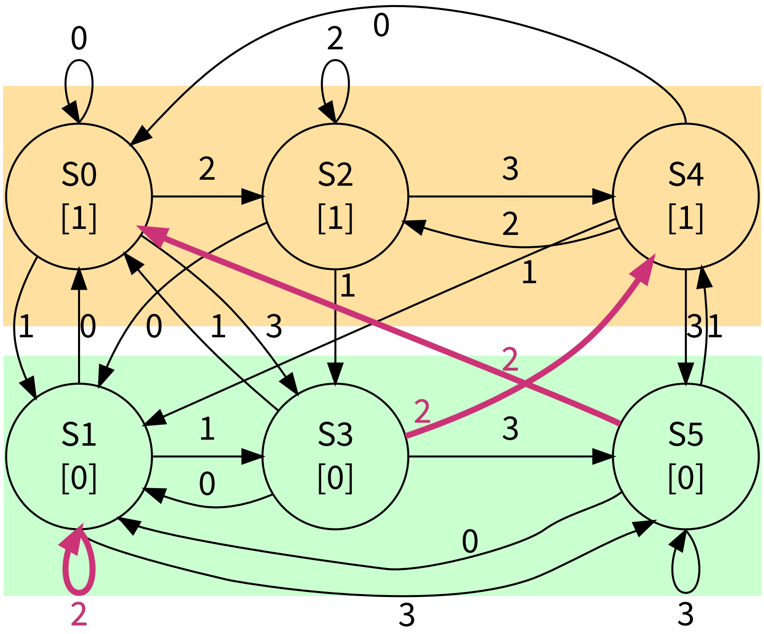

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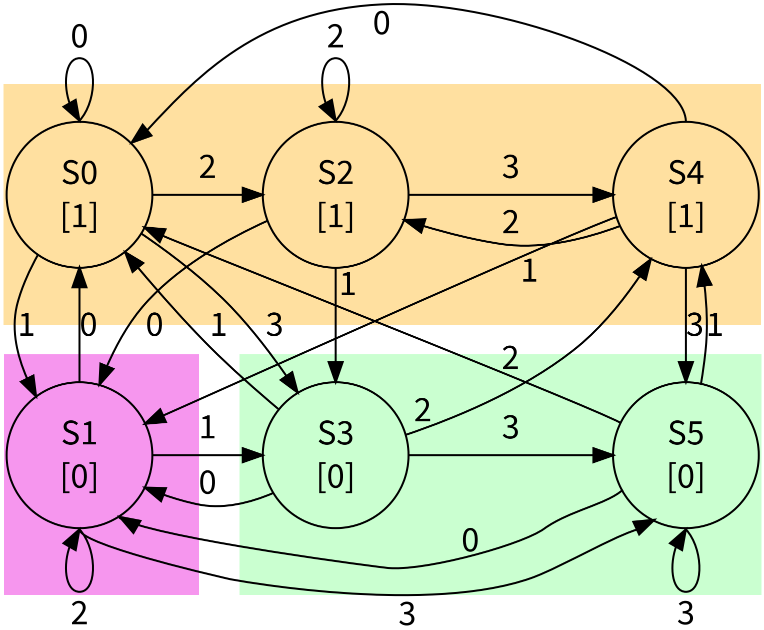

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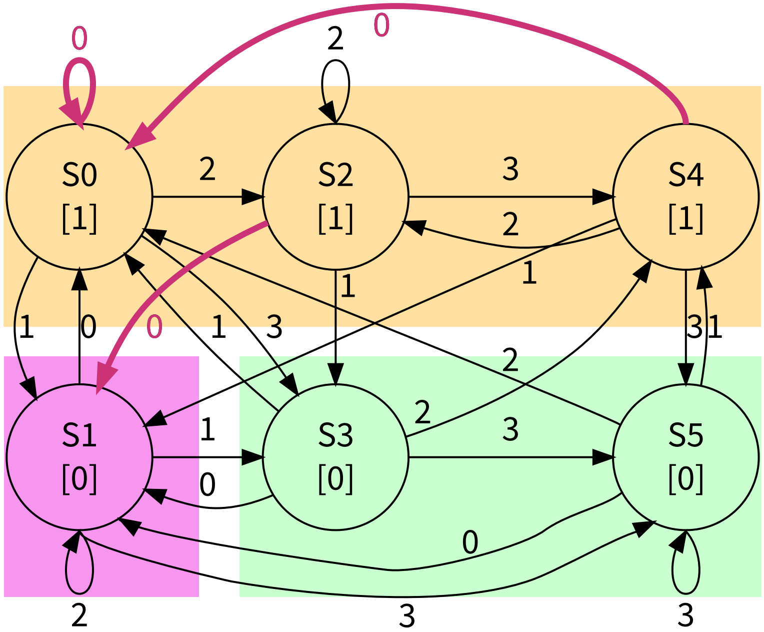

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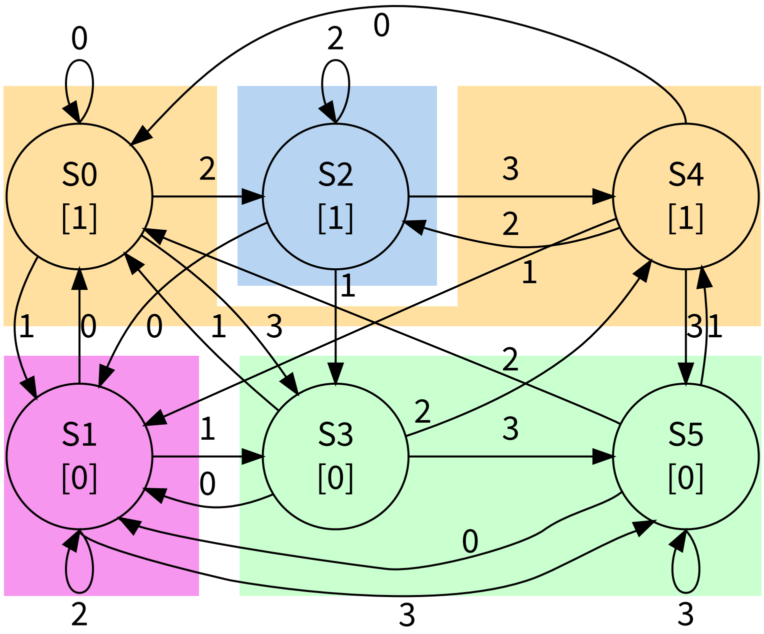

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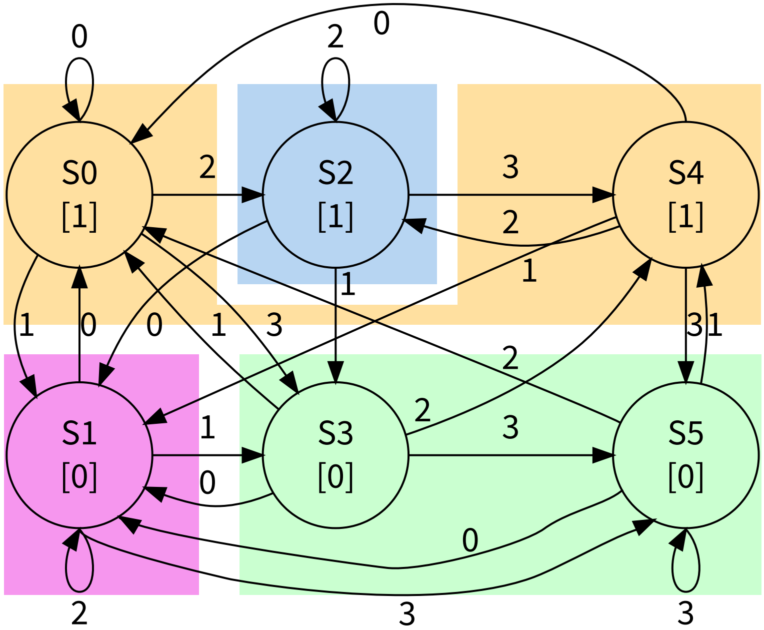

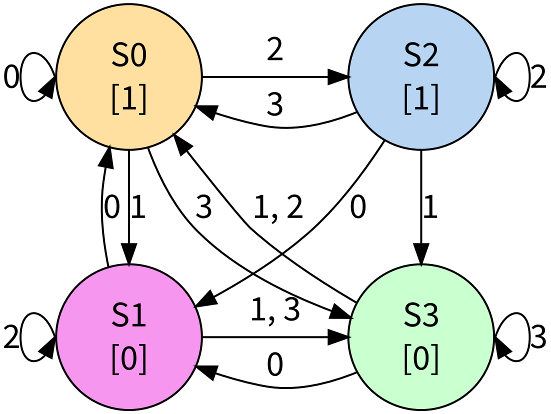

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Nondeterministic finite automata (NFAs)

Definition and examples.

Recall the definition of a DFA

Deterministic finite automaton (DFA)

A *deterministic finite automaton* (DFA) $M = (S, \Sigma_{\text{in}}, f, s_0, F)$ consists of
a finite set of *states* S, a finite *input alphabet* Σ_{in},
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A string x is in the language recognized by a DFA if and only if x labels a path from the start state to some final state.

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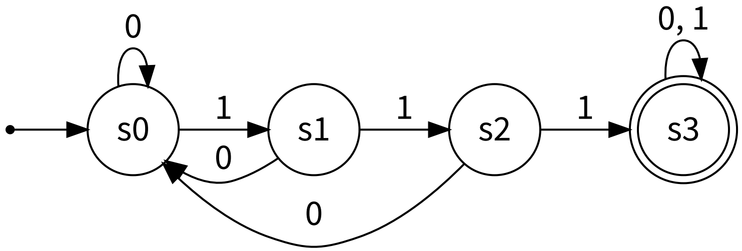

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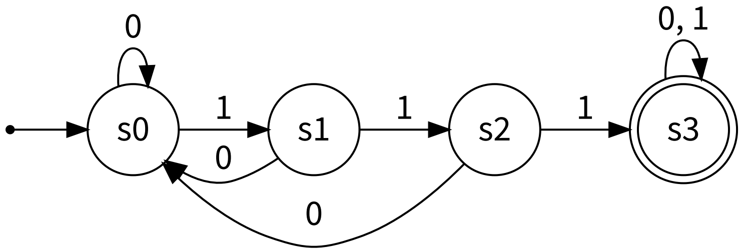

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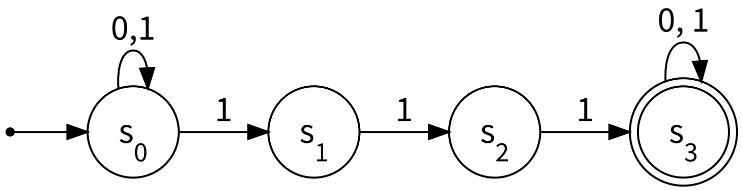

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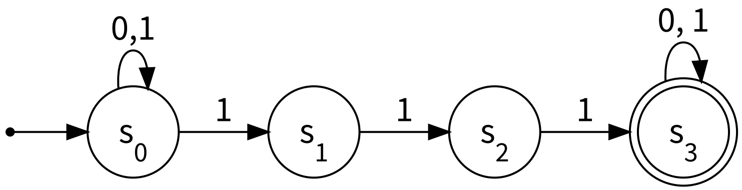

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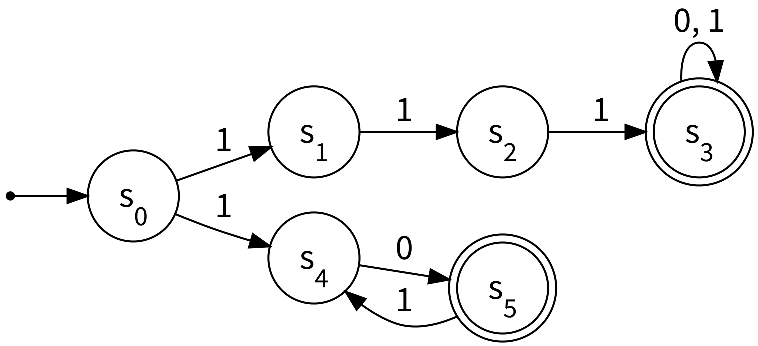
Example NFAs



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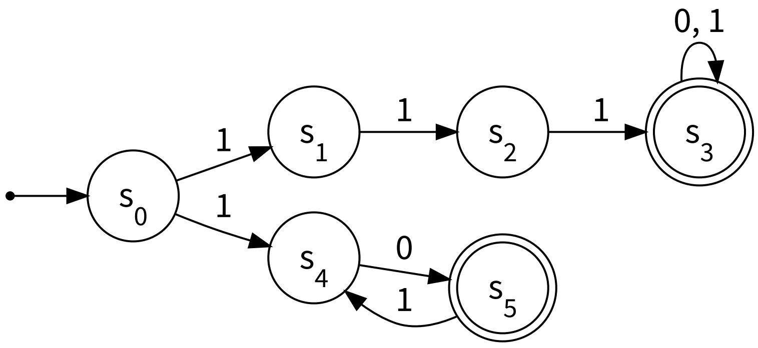

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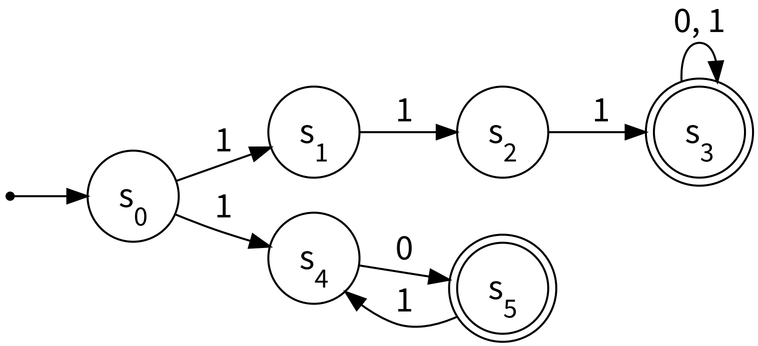

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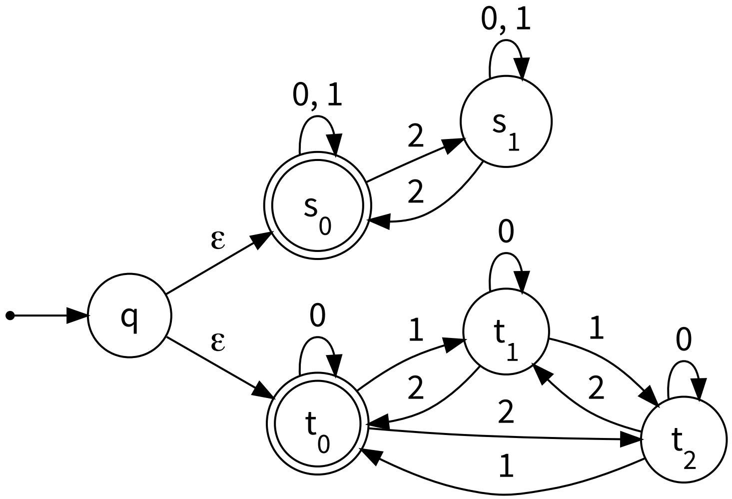

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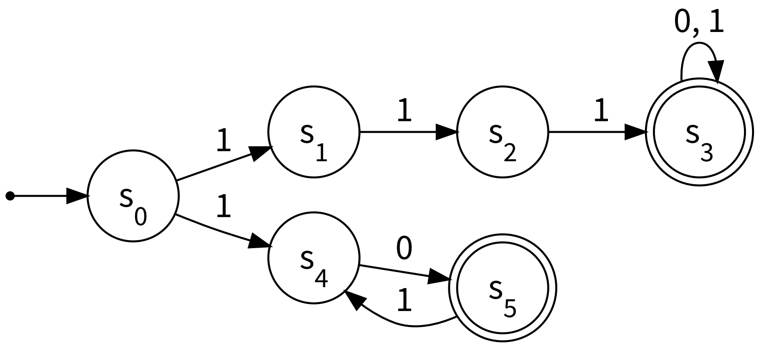

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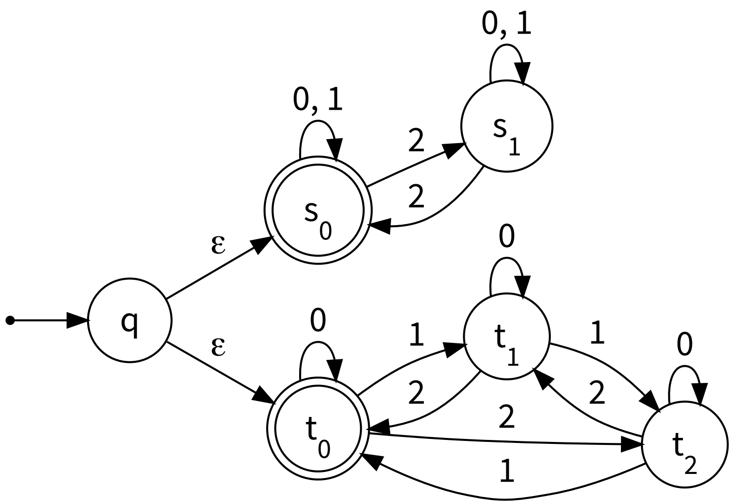
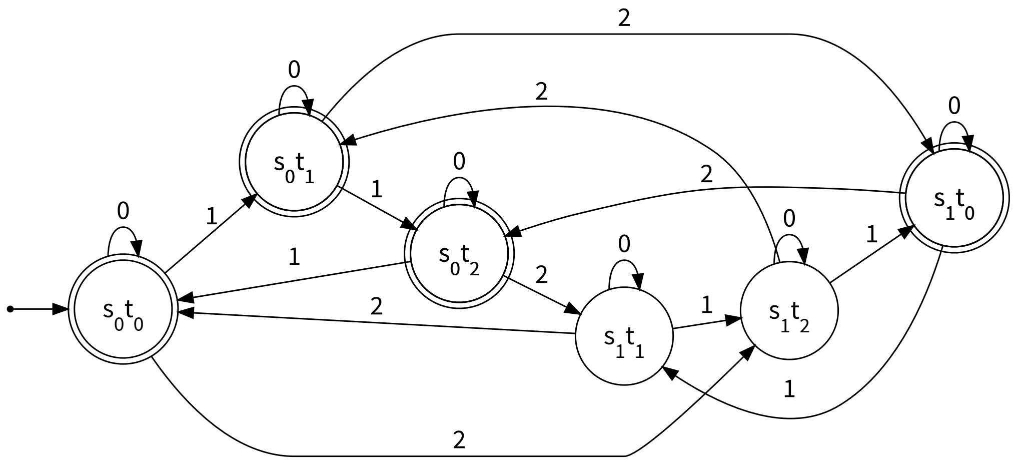
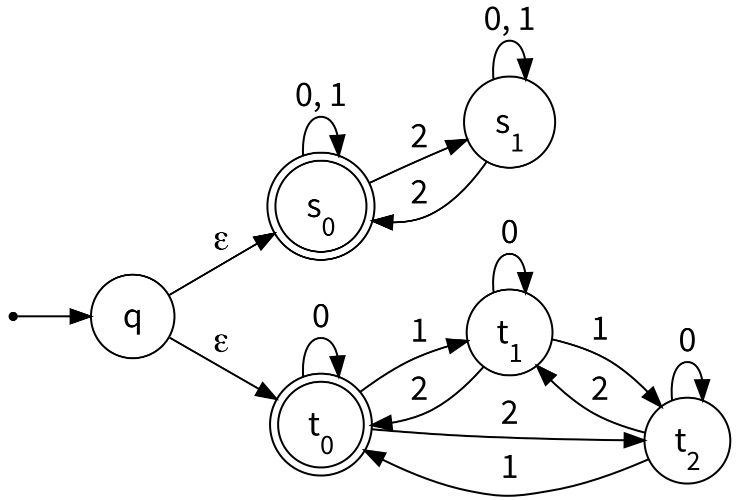


What language does this NFA accept?

Strings over {0, 1, 2} with an even number of 2's *or* with digits summing to 0 mod 3.

NFAs make it easy to union languages





A DFA and NFA for the same language.

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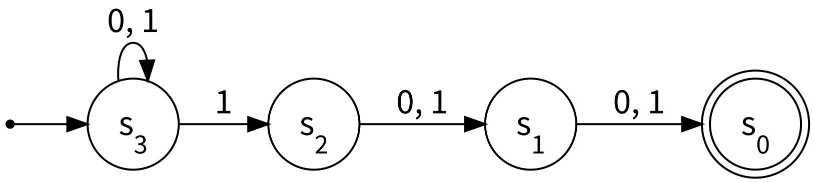
NFAs can be much smaller than DFAs



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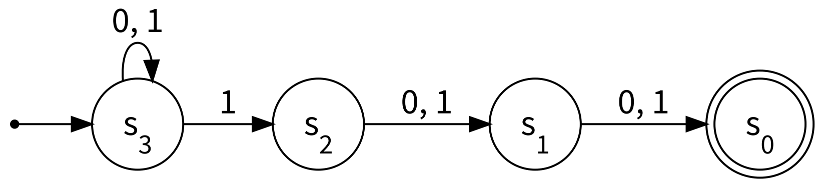

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Binary strings with a 1 in the 3rd position from the end.

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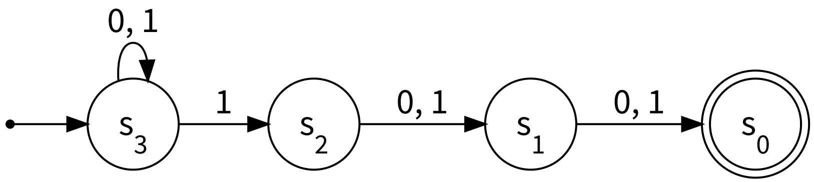

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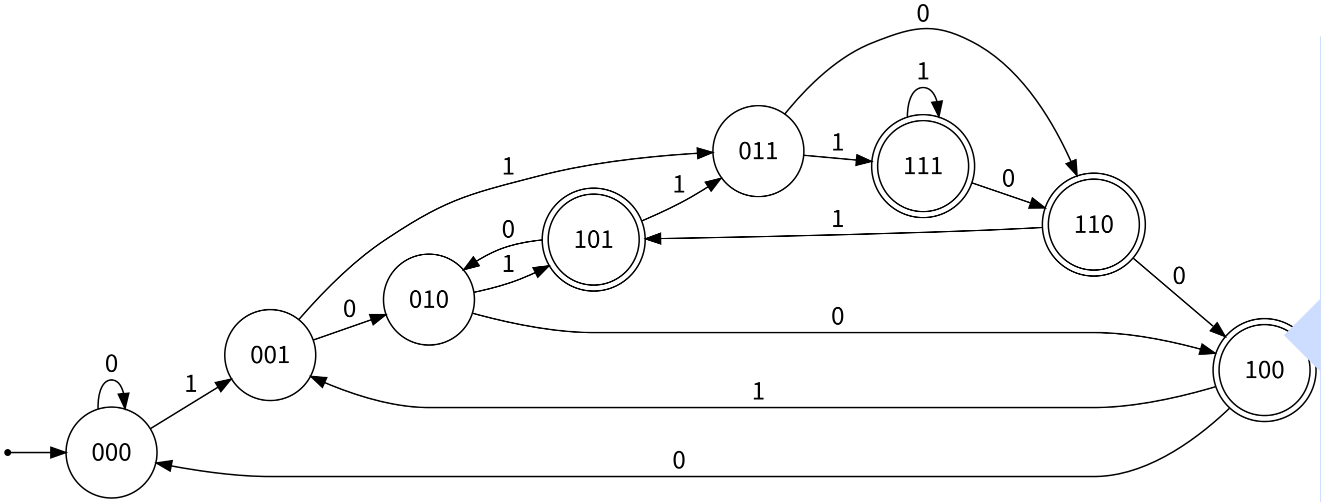


The smallest DFA that accepts the same language.

Three ways to understand NFAs

Outside observer

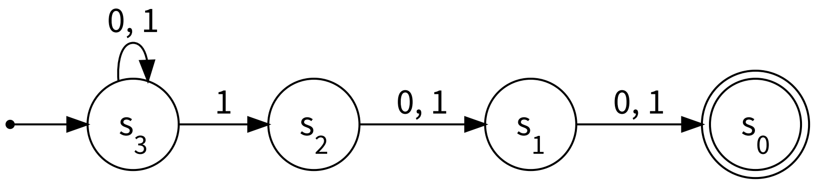
Is there a path labeled by x from the start state to some final state?

Perfect guesser (oracle)

Given an input x, the NFA guesses the right edge to take (if one exists) whenever there is a choice to be made.

Parallel exploration

The NFA runs all possible computations on x in parallel.



Summary

Every FSM has a unique minimal equivalent FSM (modulo state names).

We can compute the minimal FSM using the algorithm from this lecture.

We can use this to check if two FSMs are equivalent!

An NFA recognizes a set of strings (language).

An NFA differs from a DFA in that the transition function maps each state and input symbol to a *set* of states.

Determining if an NFA accepts string boils down to checking if there is a path from the start state to some final state, where the path consists of the edges labeled by the string's characters.