



CSE 311 Lecture 22: Relations and Directed Graphs

Emina Torlak and Sami Davies

Topics

Relations

Relations, properties, operations, and applications.

Directed graphs

Directed graphs and representing relations as directed graphs.

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What are relations?

Binary relation

Let A and B be sets.

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A *binary relation* from A to B is a subset of $A \times B$.

A binary relation R is a set of tuples. If R is a relation from a set A to itself, we say that R is a relation **on** A .

Examples of relations we've seen before

\geq on \mathbb{N}

$$\{(x, y) : x \geq y \text{ and } x, y \in \mathbb{N}\}$$

$<$ on \mathbb{R}

$$\{(x, y) : x < y \text{ and } x, y \in \mathbb{R}\}$$

$=$ on Σ^*

$$\{(x, y) : x = y \text{ and } x, y \in \Sigma^*\}$$

\subseteq on $\mathcal{P}(U)$ for a universe U

$$\{(A, B) : A \subseteq B \text{ and } A, B \in \mathcal{P}(U)\}$$

More examples of relations

$$R_1 = \{(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)\}$$

$$R_2 = \{(x, y) : x \equiv y \pmod{5}\}$$

$$R_3 = \{(c_1, c_2) : c_1 \text{ is a prerequisite for } c_2\}$$

$$R_4 = \{(s, c) : \text{student } s \text{ has taken course } c\}$$

Important properties of relations

Relation properties

Let R be a relation on A .

R is *reflexive* iff $(a, a) \in R$ for every $a \in A$.

R is *symmetric* iff $(a, b) \in R$ implies $(b, a) \in R$.

R is *antisymmetric* iff $(a, b) \in R$ and $a \neq b$ implies $(b, a) \notin R$.

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$\{(1, 2), (2, 1), (1, 3)\}$	none!

Operations on relations

Relation composition

Let R be a relation on A to B .

Let S be a relation on B to C .

The *composition* of R and S , $R \circ S$, is the relation from A to C defined by:

$$R \circ S = \{(a, c) : \exists b. (a, b) \in R \text{ and } (b, c) \in S\}$$

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Since relations are just sets, we can also combine them with the standard set theoretic operators (\cup , \cap , \setminus).

Examples of relational composition

$$R \circ S = \{(a, c) : \exists b. (a, b) \in R \text{ and } (b, c) \in S\}$$

Consider the relations Parent and Sister.

$(a, b) \in \text{Parent}$ iff b is a parent of a

$(a, b) \in \text{Sister}$ iff b is a sister of a

When is $(x, y) \in \text{Parent} \circ \text{Sister}$?

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y is an aunt of x .

When is $(x, y) \in \text{Sister} \circ \text{Parent}$?

y is a parent of x and x has a sister.

More examples of relational composition

$$R \circ S = \{(a, c) : \exists b. (a, b) \in R \text{ and } (b, c) \in S\}$$

Use the relations Parent, Child, Sister, Brother, Sibling to express

$(a, b) \in \text{Uncle}$ iff b is an uncle of a

$(a, b) \in \text{Cousin}$ iff b is a cousin of a

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$(a, b) \in \text{Uncle}$ iff b is an uncle of a **Parent \circ Brother**

$(a, b) \in \text{Cousin}$ iff b is a cousin of a **Parent \circ Sibling \circ Child**

Powers of a relation

$$R^0 = \{(a, a) : a \in A\}$$

The *identity* relation on A .

$$R^1 = R^0 \circ R = R$$

$$R^2 = R^1 \circ R = R \circ R = \{(a, c) : \exists b. (a, b) \in R \text{ and } (b, c) \in R\}$$

⋮

$$R^{n+1} = R^n \circ R \text{ for } n \geq 0$$

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Reflexive transitive closure

Let R be a relation on A .

The *reflexive transitive closure*, R^* , of R is defined by

$$R^* = \bigcup_{n=0}^{\infty} R^n = R^0 \cup R^1 \cup R^2 \cup \dots$$

Applications of relations

Databases use relations to store and organize data.

Relations are also used to reason about software systems.

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So far, we've only talked about binary relations.

But the same concepts extend to *n-ary relations*, which contain tuples of length $n \geq 1$.

A (relational) database table is an *n-ary relation*, e.g.,

$R \subseteq \text{Student} \times \text{ID} \times \text{GPA}$.

Student	ID	GPA
Einstein	299792458	2.11
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Relations are also used to reason about software systems.

Relational logic (relations + predicate logic) is a language for specifying and **automatically checking** properties of software systems.

Many applications, including verifying safety critical software, synthesizing memory consistency models, and verifying security and privacy policies.

A filesystem spec

$\text{Root} = \{(\text{root})\}$

$\text{Root} \subseteq \text{Dir}$

$\text{contents} \subseteq \text{Dir} \times (\text{File} \cup \text{Dir})$

$(\text{File} \cup \text{Dir}) \subseteq \text{Root} \circ \text{contents}^*$

Directed graphs

Directed graphs and representing relations as directed graphs.

What is a directed graph?

Directed graphs

A *directed graph* $G = (V, E)$ consists of a set of *vertices* V and a set of *edges* $E \subseteq V \times V$, which are ordered pairs of vertices.

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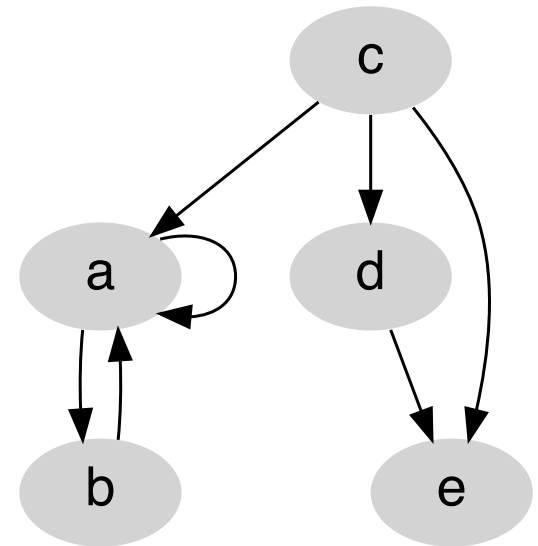
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A directed graph $G = (V, E)$ consists of a set of vertices V and a set of edges $E \subseteq V \times V$, which are ordered pairs of vertices.

Example directed graph $G = (V, E)$

$V = \{a, b, c, d, e\}$

$E = \{(a, a), (a, b), (b, a), (c, a), (c, d), (c, e), (d, e)\}$



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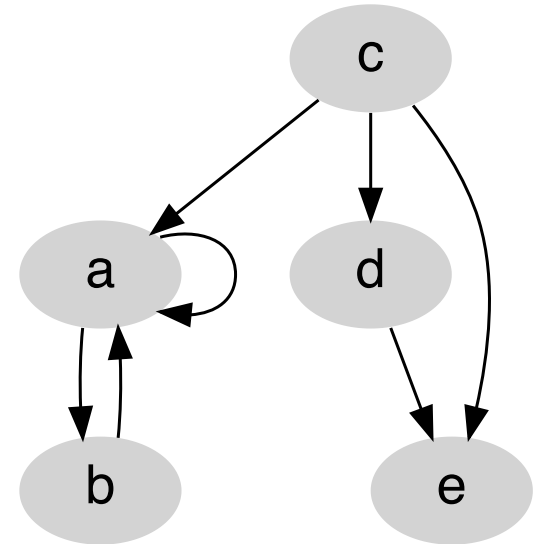
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E is just a relation on V !

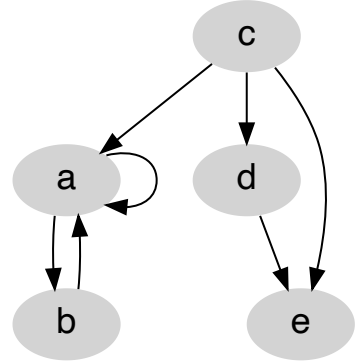


Representing relations as directed graphs

A relation R on A corresponds to the graph $G = (A, R)$.

$$R = \{(a, a), (a, b), (b, a), (c, a), (c, d), (c, e), (d, e)\}$$

$$A = \{a, b, c, d, e\}$$

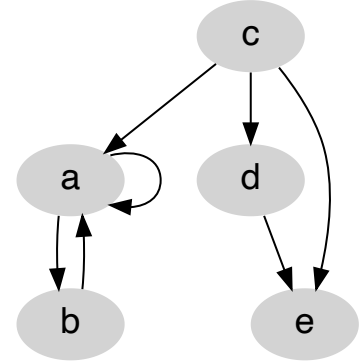


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How about a relation R from A to B ?

$$R = \{(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)\}$$

$$A = \{a, b, c\}$$

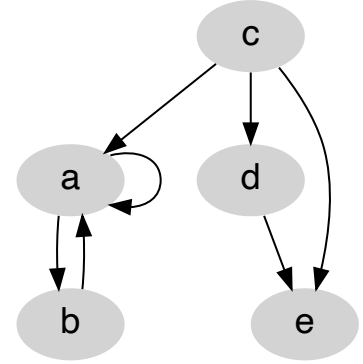
$$B = \{1, 2, 3\}$$

Representing relations as directed graphs

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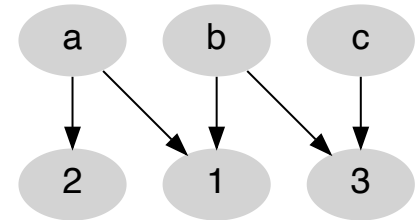
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$$A = \{a, b, c\}$$

$$B = \{1, 2, 3\}$$

$$G = (A \cup B, R)$$



Paths in graphs

Path in a directed graph

Let $G = (V, E)$ be a directed graph.

A *path* in G is a sequence of vertices v_0, v_1, \dots, v_k where $(v_i, v_{i+1}) \in E$ for each $0 \leq i < k$.

Paths in graphs

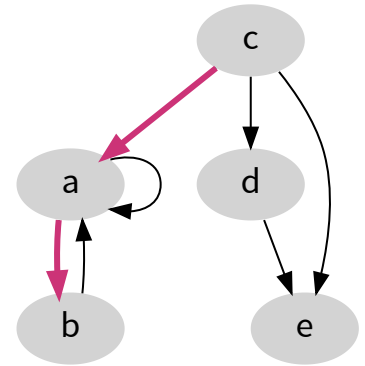
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Example paths

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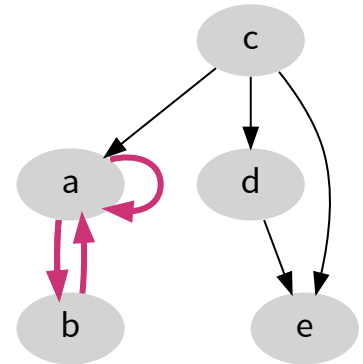
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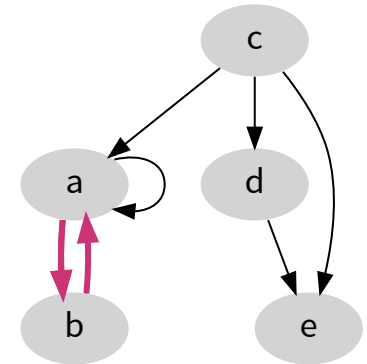
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Paths in graphs and relations

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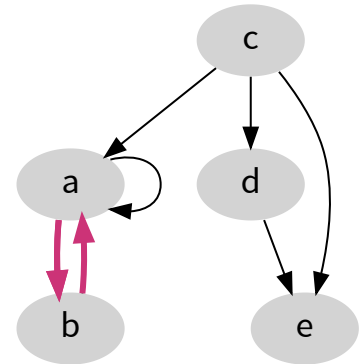
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Cycle ($v_0 = v_k$): b, a, a, a, b

Simple cycle ($v_0 = v_k$ and no other v_i repeated): b, a, b



We've defined paths on graphs but the same definition applies to relations, since a relation and its graph representation are interchangeable.

Relational composition using directed graphs

If R and S are relations on A , compute $R \circ S$ as follows:

Create the digraph $G = (A, R \cup S)$.

Add an edge (a, b) to $R \circ S$ iff there is a path a, v, b in G such that $(a, v) \in R$ and $(v, b) \in S$.

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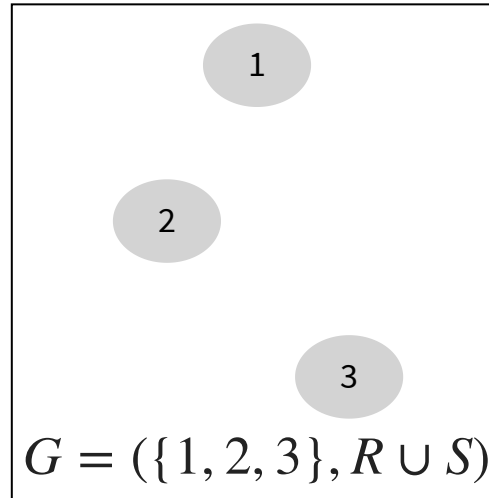
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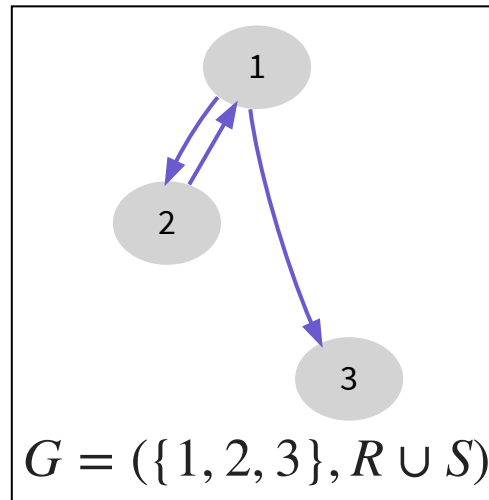
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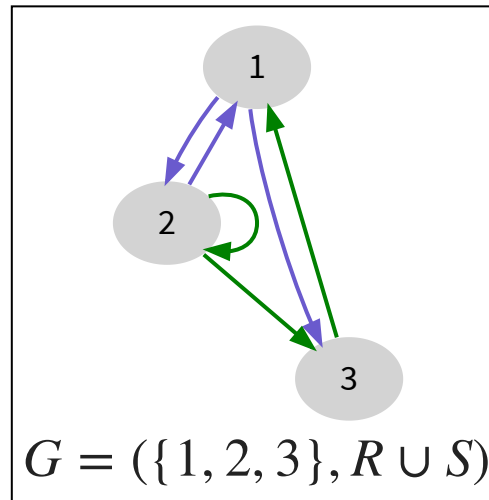
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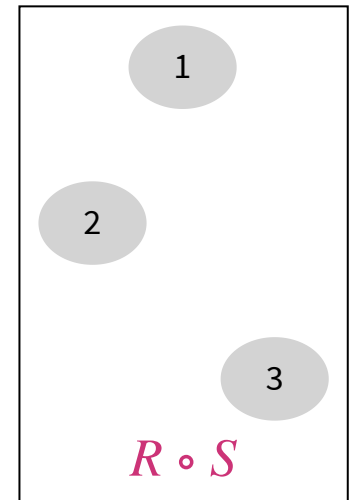
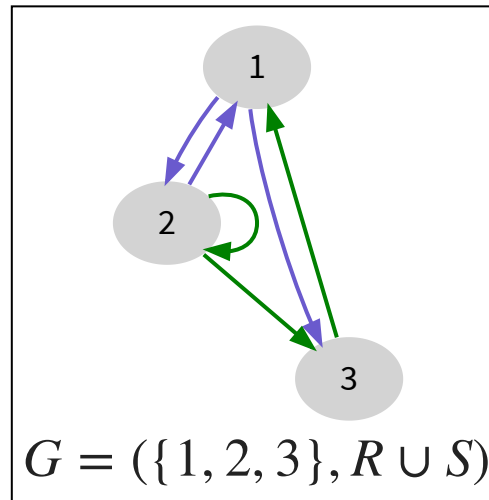
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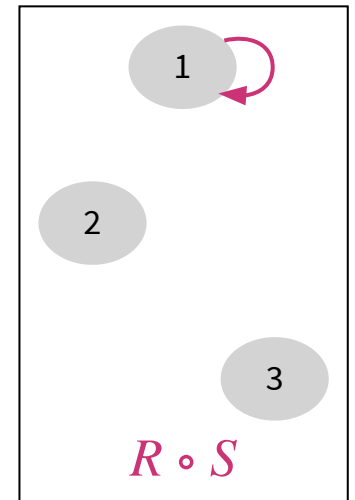
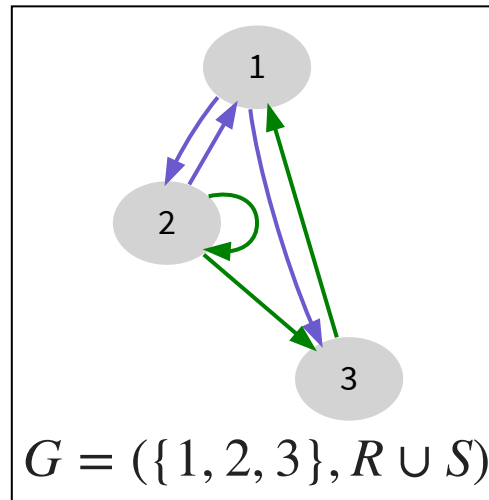
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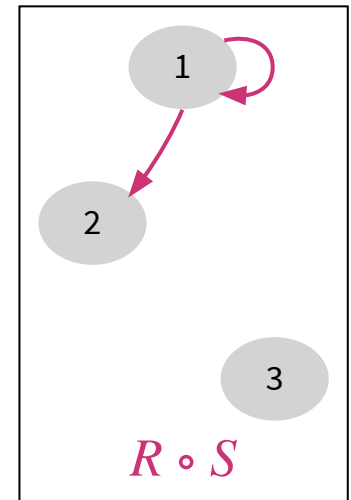
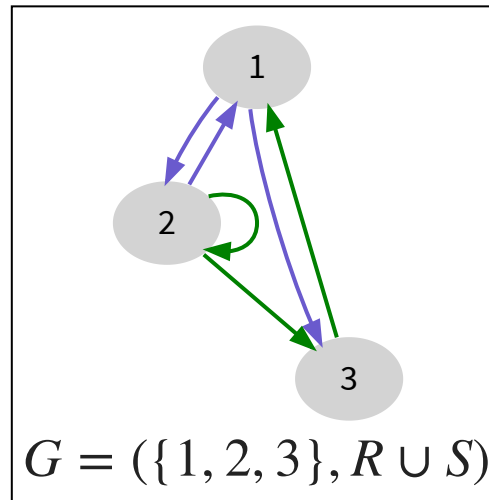
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Relational composition using directed graphs

If R and S are relations on A , compute $R \circ S$ as follows:

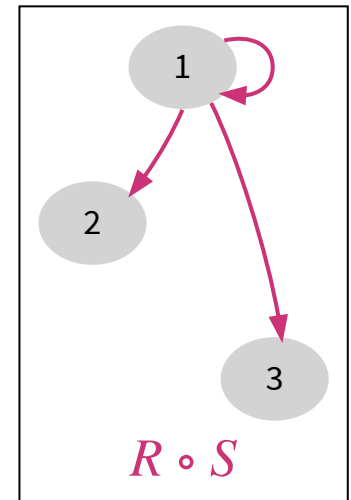
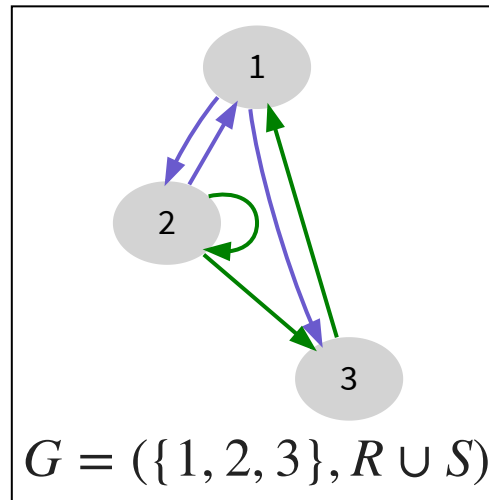
Create the digraph $G = (A, R \cup S)$.

Add an edge (a, b) to $R \circ S$ iff there is a path a, v, b in G such that $(a, v) \in R$ and $(v, b) \in S$.

Example:

$$R = \{(1, 2), (2, 1), (1, 3)\}$$

$$S = \{(2, 2), (2, 3), (3, 1)\}$$



Powers of a relation using directed graphs

Length of a path in a graph

The *length* of a path v_0, \dots, v_k is the number of edges in it, i.e., k .

Powers and paths

Let R be a relation on a set A . There is a path of length n from a to b in R if and only if $(a, b) \in R^n$.

Reflexive transitive closure using directed graphs

Connectivity relation

Let R be a relation on a set A . The *connectivity* relation R^\star consists of the tuples (a, b) such that there is a path (of any length) from a to b in R .

Connectivity and reflexive transitive closure

The reflexive transitive closure R^\star of a relation R is its connectivity relation R^\star , i.e., $R^\star = R^* = \bigcup_{n=0}^{\infty} R^n$.

Summary

Relations are a fundamental structure in computer science.

A relation is a set of tuples.

Relations can be reflexive, (anti)symmetric, transitive.

We can combine binary relations with the composition \circ operator.

Directed graphs consist of nodes and edges (ordered pairs of nodes).

Relations can be represented as directed graphs.

The two representations are interchangeable: relational operations have corresponding graph operations and vice versa.