CSE 311 Lecture 19: Structural Induction

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Topics

Recursively defined sets

A brief review of Lecture 18.

Structural induction

A method for proving properties of recursive structures.

Using structural induction

Example proofs about recursively defined numbers, strings, and trees.

Recursively defined sets

A brief review of Lecture 18.

Giving a recursive definition of a set

A recursive definition of a set S has the following parts:

Basis step specifies one or more initial members of S.

Recursive step specifies the rule(s) for constructing new elements of S from the existing elements.

Exclusion (or closure) rule states that every element in S follows from the basis step and a finite number of recursive steps.

The exclusion rule is assumed, so no need to state it explicitly.

Recursively strings and functions on them

Let Σ be a finite set of characters, and define Σ^* to be the set of of all strings over Σ:

Basis: $\varepsilon \in \Sigma^*$, where ε is the empty string.

Recursive: if $w \in \Sigma^*$ and $a \in \Sigma$, then $wa \in \Sigma^*$

Length

$\mathrm{len}(\varepsilon) = 0$

$\mathrm{len}(wa) = \mathrm{len}(w) + 1$ for $w \in \Sigma^*, a \in \Sigma$

Reversal

$\varepsilon^{\mathrm{R}} = \varepsilon$

$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*, a \in \Sigma$

Concatenation

$x \cdot \varepsilon = x$ for $x \in \Sigma^*$

$x \cdot (wa) = (x \cdot w)a$ for $x, w \in \Sigma^*, a \in \Sigma$

Number of c's in a string

$\#_c(\varepsilon) = 0$

$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma, a \neq c$

Rooted binary trees and functions on them

Rooted binary trees

Basis: $\cdot \in S$

Recursive: if $L \in S$ and $R \in S$, then $\mathrm{Tree}(\cdot, L, R) \in S$

Size of a rooted binary tree

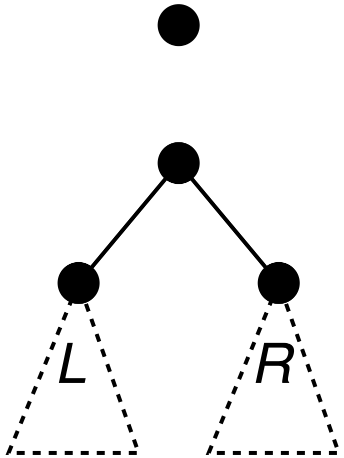
$|\cdot| = 1$

$|\mathrm{Tree}(\cdot, L, R)| = 1 + |L| + |R|$

Height of a rooted binary tree

$\lceil \cdot \rceil = 0$

$\lceil \mathrm{Tree}(\cdot, L, R) \rceil = 1 + \max(\lceil L \rceil, \lceil R \rceil)$



Structural induction

A method for proving properties of recursive structures.

How can we prove properties of recursive structures?

Suppose that S is a recursively defined set.

And we want to prove that every element of S satisfies a predicate P.

Can we use ordinary induction to prove $\forall x \in S.\ P(x)$?

How can we prove properties of recursive structures?

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And we want to prove that every element of S satisfies a predicate P.

Can we use ordinary induction to prove $\forall x \in S.\, P(x)$?

Yes! Define $Q(n)$ to be “for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true.”

How can we prove properties of recursive structures?

Suppose that S is a recursively defined set.

And we want to prove that every element of S satisfies a predicate P.

Can we use ordinary induction to prove $\forall x \in S.\ P(x)$?

Yes! Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

But this proof would be long and cumbersome to do!

So we use **structural induction** instead.

- Follows from ordinary induction (on Q), while providing a more convenient proof template for reasoning about recursive structures.
- As powerful as ordinary induction, which is just structural induction applied to the recursively defined set of natural numbers.

Proving $\forall x \in S.\ P(x)$ by structural induction

① **Let $P(x)$ be** *[definition of P(x)]*.

We will show that $P(x)$ is true for every $x \in S$ by structural induction.

② **Base cases:**

[Proof of $P(s_0)$, …, $P(s_m)$.]

③ **Inductive hypothesis:**

Assume that $P(y_0), \ldots, P(y_k)$ are true for some arbitrary $y_0, \ldots, y_k \in S$.

④ **Inductive step:**

We want to prove that $P(y)$ is true.

[Proof of P(y). The proof ***must*** *invoke the structural inductive hypothesis.]*

⑤ **The result follows for all $x \in S$ by structural induction.**

Recursive definition of S

Basis step:

$s_0 \in S, \ldots, s_m \in S.$

Recursive step:

if $y_0, \ldots, y_k \in S$, then $y \in S$.

Proving $\forall x \in S.\ P(x)$ by structural induction

① **Let $P(x)$ be** *[definition of $P(x)$]*.

We will show that $P(x)$ is true for every $x \in S$ by structural induction.

② **Base cases:**

[Proof of $P(s_0), \ldots, P(s_m)$.]

③ **Inductive hypothesis:**

Assume that $P(y_0), \ldots, P(y_k)$ are true for some arbitrary $y_0, \ldots, y_k \in S$.

④ **Inductive step:**

We want to prove that $P(y)$ is true.

*[Proof of $P(y)$. The proof **must** invoke the structural inductive hypothesis.]*

⑤ **The result follows for all $x \in S$ by structural induction.**

Recursive definition of S

Basis step:

$s_0 \in S, \ldots, s_m \in S.$

Recursive step:

if $y_0, \ldots, y_k \in S$, then $y \in S$.

If the **recursive step** of S includes multiple rules for constructing new elements from existing elements, then ③ **assume** P for the existing elements in every rule, and ④ **prove** P for the new element in every rule.

Structural induction works just like ordinary induction

① **Let $P(x)$ be** *[definition of P(x)]*.

We will show that $P(x)$ is true for every $x \in \mathbb{N}$ by structural induction.

② **Base cases:**

[Proof of $P(0)$.]

③ **Inductive hypothesis:**

Assume that $P(n)$ is true for some arbitrary $n \in \mathbb{N}$.

④ **Inductive step:**

We want to prove that $P(n+1)$ is true.

[Proof of $P(n+1)$. The proof ***must*** *invoke the structural inductive hypothesis.]*

⑤ **The result follows for all $x \in \mathbb{N}$ by structural induction.**

Recursive definition of $\mathbb{N}$

Basis step: $0 \in \mathbb{N}$.

Recursive step:

if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$.

Ordinary induction is just structural induction applied to the recursively defined set of natural numbers!

Understanding structural induction

$$P(\cdot); \forall L, R \in S.\ (P(L) \wedge P(R)) \rightarrow P(\mathrm{Tree}(\cdot, L, R))$$
$$\therefore \forall x \in S.\ P(x)$$

Define S by
Basis: $\cdot \in S$.
Recursive:
if $L, R \in S$, then
$\mathrm{Tree}(\cdot, L, R) \in S$

How do we get $P(\mathrm{Tree}(\cdot, \cdot, \mathrm{Tree}(\cdot, \cdot, \cdot)))$ from $P(\cdot)$ and $\forall L, R \in S.\ (P(L) \wedge P(R)) \rightarrow P(\mathrm{Tree}(\cdot, L, R))$?

1. First, we have $\forall L, R \in S.\ (P(L) \wedge P(R)) \rightarrow P(\mathrm{Tree}(\cdot, L, R))$
2. Next, we have $P(\cdot)$.
3. Intro $\wedge$ on 2 gives us $P(\cdot) \wedge P(\cdot)$.
4. Elim $\forall$ on 1 gives us $(P(\cdot) \wedge P(\cdot)) \rightarrow P(\mathrm{Tree}(\cdot, \cdot, \cdot))$.
5. Modus Ponens on 3 and 4 gives us $P(\mathrm{Tree}(\cdot, \cdot, \cdot))$.
6. Intro $\wedge$ on 2 and 5 gives us $P(\cdot) \wedge P(\mathrm{Tree}(\cdot, \cdot, \cdot))$.
7. Elim $\forall$ on 1 gives us $(P(\cdot) \wedge P(\mathrm{Tree}(\cdot, \cdot, \cdot)) \rightarrow P(\mathrm{Tree}(\cdot, \cdot, \mathrm{Tree}(\cdot, \cdot, \cdot)))$
   .
8. Modus Ponens on 6 and 7 gives us $P(\mathrm{Tree}(\cdot, \cdot, \mathrm{Tree}(\cdot, \cdot, \cdot)))$.

$\boldsymbol{P(\cdot)}$

$\boldsymbol{P(\cdot) \wedge P(\cdot)}$

$\Downarrow\ \boldsymbol{(P(\cdot) \wedge P(\cdot)) \rightarrow P(\mathrm{Tree}(\cdot, \cdot, \cdot))}$

$\boldsymbol{P(\mathrm{Tree}(\cdot, \cdot, \cdot))}$

$\boldsymbol{P(\cdot) \wedge P(\mathrm{Tree}(\cdot, \cdot, \cdot))}$

$\Downarrow\ \boldsymbol{(P(\cdot) \wedge P(\mathrm{Tree}(\cdot, \cdot, \cdot)) \rightarrow P(\mathrm{Tree}(\cdot, \cdot, \mathrm{Tree}(\cdot, \cdot, \cdot)}$

$\boldsymbol{P(\mathrm{Tree}(\cdot, \cdot, \mathrm{Tree}(\cdot, \cdot, \cdot)))}$

Using structural induction

Example proofs about recursively defined numbers, strings, and trees.

Prove that every $x \in S$ is divisible by 3

Define S by

Basis: $6 \in S$, $15 \in S$.

Recursive: if $x, y \in S$, then $x + y \in S$.

Prove that every $x \in S$ is divisible by 3

① **Let $P(x)$ be $3 \mid x$.**

We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Define S by

Basis: $6 \in S$, $15 \in S$.

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Prove that every $x \in S$ is divisible by 3

① **Let $P(x)$ be $3 \mid x$.**

We will show that $P(x)$ is true for every $x \in S$ by structural induction.

② **Base cases ($x = 6, x = 15$):**

$3 \mid 6$ so $P(6)$ holds, and $3 \mid 15$ so $P(15)$ holds.

Define S by

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$3 \mid 6$ so $P(6)$ holds, and $3 \mid 15$ so $P(15)$ holds.

③ **Inductive hypothesis:**

Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

Define S by

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Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

④ **Inductive step:**

We want to prove that $P(x + y)$ is true.

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Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

④ **Inductive step:**

We want to prove that $P(x + y)$ is true.

By the inductive hypothesis, $3 \mid x$ and $3 \mid y$, so $x = 3i$ and $y = 3j$ for some $i, j \in Z$.

Define S by

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Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

④ **Inductive step:**

We want to prove that $P(x + y)$ is true.
By the inductive hypothesis, $3 \mid x$ and $3 \mid y$, so $x = 3i$ and $y = 3j$ for some $i, j \in \mathbb{Z}$. Therefore,
$x + y = 3i + 3j = 3(i + j)$ so $3 \mid (x + y)$.

Define S by

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⑤ **The result follows for all $x \in S$ by structural induction.**

Define S by

Basis: $6 \in S$, $15 \in S$.

Recursive: if $x, y \in S$, then $x + y \in S$.

Prove $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$

Define Σ^* by

Basis: $\varepsilon \in \Sigma^*$.

Recursive:

if $w \in \Sigma^*$ and

$a \in \Sigma$,

then $wa \in \Sigma^*$

Length

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = \text{len}(w) + 1$

Concatenation

$x \cdot \varepsilon = x$

$x \cdot (wa) = (x \cdot w)a$

Prove $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$

What object (x or y) to do structural induction on?

Define Σ^* by

Basis: $\varepsilon \in \Sigma^*$. [14]

Recursive:

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① **Let $P(y)$ be $\forall x \in \Sigma^*.\ \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.**

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We will show that $P(y)$ is true for every $y \in \Sigma^*$ by structural induction.

② **Base case ($y = \varepsilon$):**

For every $x \in \Sigma^*$, $\mathrm{len}(x \cdot \varepsilon) = \mathrm{len}(x) = \mathrm{len}(x) + \mathrm{len}(\varepsilon)$ since $\mathrm{len}(\varepsilon) = 0$.
So $P(\varepsilon)$ is true.

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③ **Inductive hypothesis:**

Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$.

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if $w \in \Sigma^*$ and

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① **Let $P(y)$ be $\forall x \in \Sigma^*.\ \mathrm{len}(x \cdot y) = \mathrm{len}(x) + \mathrm{len}(y)$.**

We will show that $P(y)$ is true for every $y \in \Sigma^*$ by structural induction.

② **Base case ($y = \varepsilon$):**

For every $x \in \Sigma^*$, $\mathrm{len}(x \cdot \varepsilon) = \mathrm{len}(x) = \mathrm{len}(x) + \mathrm{len}(\varepsilon)$ since $\mathrm{len}(\varepsilon) = 0$.
So $P(\varepsilon)$ is true.

③ **Inductive hypothesis:**

Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$.

④ **Inductive step:**

We want to prove that $P(wa)$ is true for every $a \in \Sigma$.

Define Σ^* by

Basis: $\varepsilon \in \Sigma^*$.

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① **Let $P(y)$ be $\forall x \in \Sigma^*.\ \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.**

We will show that $P(y)$ is true for every $y \in \Sigma^*$ by structural induction.

② **Base case ($y = \varepsilon$):**

For every $x \in \Sigma^*$, $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ since $\text{len}(\varepsilon) = 0$.
So $P(\varepsilon)$ is true.

③ **Inductive hypothesis:**

Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$.

④ **Inductive step:**

We want to prove that $P(wa)$ is true for every $a \in \Sigma$.

Let $a \in \Sigma$ and $x \in \Sigma^*$ be arbitrary. Then

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① **Let $P(y)$ be $\forall x \in \Sigma^*. \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.**

We will show that $P(y)$ is true for every $y \in \Sigma^*$ by structural induction.

② **Base case ($y = \varepsilon$):**

For every $x \in \Sigma^*$, $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ since $\text{len}(\varepsilon) = 0$. So $P(\varepsilon)$ is true.

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We want to prove that $P(wa)$ is true for every $a \in \Sigma$.

Let $a \in \Sigma$ and $x \in \Sigma^*$ be arbitrary. Then

$$
\begin{aligned}
\text{len}(x \cdot wa) &= \text{len}((x \cdot w)a) && \text{by defn of } \cdot \\
&= \text{len}(x \cdot w) + 1 && \text{by defn of len} \\
&= \text{len}(x) + \text{len}(w) + 1 && \text{by IH} \\
&= \text{len}(x) + \text{len}(wa) && \text{by defn of len}
\end{aligned}
$$

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We will show that $P(y)$ is true for every $y \in \Sigma^*$ by structural induction.

② **Base case ($y = \varepsilon$):**

For every $x \in \Sigma^*$, $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ since $\text{len}(\varepsilon) = 0$.
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Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$.

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&= \text{len}(x) + \text{len}(wa) && \text{by defn of len}
\end{aligned}$$

So $\text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa)$ for all $x \in \Sigma^*$, and $P(wa)$ is true.

Define Σ^* by

Basis: $\varepsilon \in \Sigma^*$.

Recursive:

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$a \in \Sigma$,
then $wa \in \Sigma^*$

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&= \text{len}(x) + \text{len}(wa) && \text{by defn of len}
\end{aligned}$$

So $\text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa)$ for all $x \in \Sigma^*$, and $P(wa)$ is true.

⑤ **The result follows for all $y \in \Sigma^*$ by structural induction.**

Define Σ^* by

Basis: $\varepsilon \in \Sigma^*$.

Recursive:

if $w \in \Sigma^*$ and

$a \in \Sigma$,

then $wa \in \Sigma^*$

Length

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = \text{len}(w) + 1$

Concatenation

$x \cdot \varepsilon = x$

$x \cdot (wa) = (x \cdot w)a$

Prove $|t| \leq 2^{\lceil t \rceil + 1} - 1$ for every rooted binary tree t

Define S **by**

Basis: $\cdot \in S$.

Recursive:

if $L, R \in S$, then

$\text{Tree}(\cdot, L, R) \in S$

Size

$|\cdot| = 1$

$|\text{Tree}(\cdot, L, R)| =$

$1 + |L| + |R|$

Height

$\lceil \cdot \rceil = 0$

$\lceil \text{Tree}(\cdot, L, R)) \rceil$

$1 + \max(\lceil L \rceil,$

Prove $|t| \leq 2^{\lceil t \rceil + 1} - 1$ for every rooted binary tree t

① **Let $P(t)$ be $|t| \leq 2^{\lceil t \rceil + 1} - 1$.**

We will show that $P(t)$ is true for every $t \in S$ by structural induction.

Define S by

Basis: $\cdot \in S$.

Recursive:

if $L, R \in S$, then

$\mathrm{Tree}(\cdot, L, R) \in S$

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① **Let $P(t)$ be $|t| \leq 2^{\lceil t \rceil + 1} - 1$.**

We will show that $P(t)$ is true for every $t \in S$ by structural induction.

② **Base case ($t = \cdot$):**

$|\cdot| = 1 = 2^1 - 1 = 2^{0+1} - 1 = 2^{\lceil \cdot \rceil + 1} - 1$ so $P(\cdot)$ is true.

Define S by

Basis: $\cdot \in S$.

Recursive:

if $L, R \in S$, then

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③ **Inductive hypothesis:**

Assume that $P(L)$ and $P(R)$ are true for some arbitrary $L, R \in S$.

Define S by

Basis: $\cdot \in S$.

Recursive:

if $L, R \in S$, then

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$|\cdot| = 1$

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③ **Inductive hypothesis:**

Assume that $P(L)$ and $P(R)$ are true for some arbitrary $L, R \in S$.

④ **Inductive step:**

We want to prove that $P(\text{Tree}(\cdot, L, R))$ is true.

Define S by

Basis: $\cdot \in S$.

Recursive:

if $L, R \in S$, then

$\text{Tree}(\cdot, L, R) \in S$

Size

$|\cdot| = 1$

$|\text{Tree}(\cdot, L, R)| =$

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Assume that $P(L)$ and $P(R)$ are true for some arbitrary $L, R \in S$.

④ **Inductive step:**

We want to prove that $P(\text{Tree}(\cdot, L, R))$ is true.

$$
\begin{aligned}
|\text{Tree}(\cdot, L, R)| &= 1 + |L| + |R| && \text{by defn of } |\,| \\
&\leq 1 + (2^{\lceil L \rceil + 1} - 1) + (2^{\lceil R \rceil + 1} - 1) && \text{by IH} \\
&\leq 2^{\lceil L \rceil + 1} + 2^{\lceil R \rceil + 1} - 1 && \text{algebra} \\
&\leq 2^{\max(\lceil L \rceil, \lceil R \rceil) + 1} + 2^{\max(\lceil L \rceil, \lceil R \rceil) + 1} - 1 && \text{by defn of } \max \\
&\leq 2(2^{\max(\lceil L \rceil, \lceil R \rceil) + 1}) - 1 && \text{algebra} \\
&= 2(2^{\lceil \text{Tree}(\cdot, L, R) \rceil}) - 1 && \text{by defn of } \lceil\,\rceil \\
&= 2^{\lceil \text{Tree}(\cdot, L, R) \rceil + 1} - 1 && \text{which is the desired result.}
\end{aligned}
$$

Define S by

Basis: $\cdot \in S$.

Recursive:

if $L, R \in S$, then

$\text{Tree}(\cdot, L, R) \in$

Size

$|\cdot| = 1$

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Height

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Prove $|t| \leq 2^{\lceil t \rceil + 1} - 1$ for every rooted binary tree t

① **Let $P(t)$ be $|t| \leq 2^{\lceil t \rceil + 1} - 1$.**

We will show that $P(t)$ is true for every $t \in S$ by structural induction.

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Assume that $P(L)$ and $P(R)$ are true for some arbitrary $L, R \in S$.

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We want to prove that $P(\text{Tree}(\cdot, L, R))$ is true.

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\begin{aligned}
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&\leq 2^{\lceil L \rceil + 1} + 2^{\lceil R \rceil + 1} - 1 && \text{algebra} \\
&\leq 2^{\max(\lceil L \rceil, \lceil R \rceil) + 1} + 2^{\max(\lceil L \rceil, \lceil R \rceil) + 1} - 1 && \text{by defn of } \max \\
&\leq 2(2^{\max(\lceil L \rceil, \lceil R \rceil) + 1}) - 1 && \text{algebra} \\
&= 2(2^{\lceil \text{Tree}(\cdot, L, R) \rceil}) - 1 && \text{by defn of } \lceil\rceil \\
&= 2^{\lceil \text{Tree}(\cdot, L, R) \rceil + 1} - 1 && \text{which is the desired result.}
\end{aligned}
$$

⑤ **The result follows for all $t \in S$ by structural induction.**

Define S by

Basis: $\cdot \in S$.

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Summary

To define a set recursively, specify its basis and recursive step.

Recursive set definitions assume the *exclusion rule*.

We use recursive functions to operate on elements of recursive sets.

Use structural induction to prove properties of recursive structures.

Structural induction follows from ordinary induction but is easier to use.

To prove $\forall x \in S.\ P(x)$ using structural induction:

Show that P holds for the elements in the basis step of S.

Assume P for every existing element of S named in the recursive step.

Prove P for every new element of S created in the recursive step.