CSE 311 Lecture 16: Induction

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Topics

Mathematical induction

A brief review of Lecture 15.

Example proofs by induction

Example proofs about sums and divisibility.

Induction starting at any integer

Proving theorems about all integers $n \geq b$ for some $b \in \mathbb{Z}$.

Induction with a stricter hypothesis

Proving theorems by induction on a stricter hypothesis.

Mathematical induction

A brief review of Lecture 15.

What is induction?

Induction $$\frac{P(0);\ \forall k.\ P(k) \rightarrow P(k+1)}{\therefore\ \forall n.\ P(n)}$$

Domain: natural numbers ($\mathbb{N}$).

Induction is a logical rule of inference that applies (only) over $\mathbb{N}$.

If we know that a property P holds for 0, and
we know that $\forall k.\ P(k) \rightarrow P(k+1)$, then
we can conclude that P holds for all natural numbers.

Induction: how does it work?

$$\textbf{Induction}\ \frac{P(0);\ \forall k.\ P(k) \rightarrow P(k+1)}{\therefore\ \forall n.\ P(n)}$$

Domain: natural numbers ($\mathbb{N}$).

How do we get $P(5)$ from $P(0)$ and $\forall k.\ P(k) \rightarrow P(k+1)$?

1. First, we have $P(0)$.
2. Since $P(k) \rightarrow P(k+1)$ for all k, we have $P(0) \rightarrow P(1)$.
3. Applying Modus Ponens to 1 and 2, we get $P(1)$.
4. Since $P(k) \rightarrow P(k+1)$ for all k, we have $P(1) \rightarrow P(2)$.
5. Applying Modus Ponens to 3 and 4, we get $P(2)$.

$\vdots$

11. Applying Modus Ponens to 9 and 10, we get $P(5)$.

$$P(0) \overset{P(0)\rightarrow P(1)}{\Downarrow} P(1) \overset{P(1)\rightarrow P(2)}{\Downarrow} P(2) \overset{P(k)\rightarrow P(k+1)}{\Downarrow} P(5)$$

Using the induction rule in a formal proof: key parts

$$\text{Induction}\ \frac{P(0);\ \forall k.\ P(k) \to P(k+1)}{\therefore\ \forall n.\ P(n)}$$

1. Prove $P(0)$		Base case
2. Let $k \geq 0$ be an arbitrary integer		Inductive hypothesis
3.1. Assume that $P(k)$ is true		
3.2. $\ldots$		Inductive step
3.3. Prove $P(k+1)$ is true		
4. $P(k) \to P(k+1)$	Direct Proof Rule	Conclusion
5. $\forall k.\ P(k) \to P(k+1)$	Intro $\forall$: 2, 4	
6. $\forall n.\ P(n)$	Induction: 1, 5	

Translating to an English proof: the template

① **Let $P(n)$ be** *[definition of $P(n)$]*.

We will show that $P(n)$ is true for every integer $n \geq 0$ by induction.

② **Base case ($n = 0$):**

[Proof of $P(0)$.]

③ **Inductive hypothesis:**

Suppose that $P(k)$ is true for an arbitrary integer $k \geq 0$.

④ **Inductive step:**

We want to prove that $P(k+1)$ is true.

*[Proof of $P(k+1)$. This proof **must** invoke the inductive hypothesis somewhere.]*

⑤ **The result follows for all $n \geq 0$ by induction.**

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2. Let $k \geq 0$ be an arbitrary integer		**Inductive hypothesis**
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Example proofs by induction

Example proofs about sums and divisibility.

Prove $\sum_{i=0}^{n} i = n(n+1)/2$ for all $n \in \mathbb{N}$

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① **Let** $P(n)$ **be** $\sum_{i=0}^{n} i = 0 + 1 + \ldots + n = n(n+1)/2.$

We will show that $P(n)$ is true for every integer $n \geq 0$ by induction.

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② **Base case ($n = 0$):**

$\sum_{i=0}^{n} i = 0 = 0(0+1)/2$ so $P(0)$ is true.

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③ **Inductive hypothesis:**
Suppose that $P(k)$ is true for an arbitrary integer $k \geq 0$.

④ **Inductive step:**
We want to prove that $P(k+1)$ is true, i.e., $\sum_{i=0}^{k+1} i = (k+1)(k+2)/2$. Note that $\sum_{i=0}^{k+1} i = (\sum_{i=0}^{k} i) + (k+1) = (k(k+1)/2) + (k+1)$ by the inductive hypothesis. From this, we have that $(k(k+1)/2) + (k+1) = (k+1)(k/2+1) = (k+1)(k+2)/2$, which is exactly $P(k+1)$.

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① **Let $P(n)$ be $\sum_{i=0}^{n} i = 0 + 1 + \ldots + n = n(n+1)/2$.**
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We want to prove that $P(k+1)$ is true, i.e., $\sum_{i=0}^{k+1} i = (k+1)(k+2)/2$. Note that $\sum_{i=0}^{k+1} i = (\sum_{i=0}^{k} i) + (k+1) = (k(k+1)/2) + (k+1)$ by the inductive hypothesis.
From this, we have that $(k(k+1)/2) + (k+1) = (k+1)(k/2+1) = (k+1)(k+2)/2$, which is exactly $P(k+1)$.

⑤ **The result follows for all $n \geq 0$ by induction.**

What number divides $2^{2n} - 1$ for every $n \in \mathbb{N}$?

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Let's look at a few examples:

$$2^{2*0} - 1 = 1 - 1 = 0 = 3 * 0$$
$$2^{2*1} - 1 = 4 - 1 = 3 = 3 * 1$$
$$2^{2*2} - 1 = 16 - 1 = 15 = 3 * 5$$
$$2^{2*3} - 1 = 64 - 1 = 63 = 3 * 21$$
$$2^{2*4} - 1 = 256 - 1 = 255 = 3 * 85$$

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$$2^{2*3} - 1 = 64 - 1 = 63 = 3 * 21$$
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It looks like $3|(2^{2n} - 1)$.

Let's use induction to prove it!

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② **Base case ($n = 0$):**

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Suppose that $P(k)$ is true for an arbitrary integer $k \geq 0$.

④ **Inductive step:**

We want to prove that $P(k+1)$ is true, i.e., $3|(2^{2(k+1)} - 1)$. By inductive hypothesis, $3|(2^{2k} - 1)$ so $2^{2k} - 1 = 3j$ for some integer j. We therefore have that $2^{2(k+1)} - 1$ $= 2^{2k+2} - 1 = 4(2^{2k}) - 1 = 4(3j + 1) - 1 = 12j + 3 = 3(4j + 1)$. So $3|(2^{2(k+1)} - 1)$, which is exactly $P(k+1)$.

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① **Let $P(n)$ be $3|(2^{2n} - 1)$.**

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⑤ **The result follows for all $n \geq 0$ by induction.**

Induction starting at any integer

Proving theorems about all integers $n \geq b$ for some $b \in \mathbb{Z}$.

Changing the start line

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Then $(\forall n.\, Q(n)) \equiv (\forall n \geq b.\, P(n))$

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Use ordinary induction to prove Q**:**

Prove $Q(0) \equiv P(b)$.

Prove $(\forall k.\, Q(k) \rightarrow Q(k+1)) \equiv (\forall k \geq b.\, P(k) \rightarrow P(k+1))$.

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Use ordinary induction to prove Q:

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Prove $(\forall k.\, Q(k) \rightarrow Q(k+1)) \equiv (\forall k \geq b.\, P(k) \rightarrow P(k+1))$.

By convention, we don't define Q explicitly. Instead, we modify our proof template to account for the non-zero base case b.

Inductive proofs for any base case $b \in \mathbb{Z}$

① **Let $P(n)$ be** ***[definition of $P(n)$]*****.**

We will show that $P(n)$ is true for every integer $n \geq b$ by induction.

② **Base case ($n = b$):**

[Proof of $P(b)$.]

③ **Inductive hypothesis:**

Suppose that $P(k)$ is true for an arbitrary integer $k \geq b$.

④ **Inductive step:**

We want to prove that $P(k+1)$ is true.

[Proof of $P(k+1)$. This proof ***must*** *invoke the inductive hypothesis.]*

⑤ **The result follows for all $n \geq b$ by induction.**

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We want to prove that $P(k+1)$ is true, i.e., $3^{(k+1)} \geq (k+1)^2 + 3 = k^2 + 2k + 4$. Note that $3^{(k+1)} = 3(3^k) \geq 3(k^2 + 3)$ by the inductive hypothesis. From this we have $3(k^2 + 3) = 2k^2 + k^2 + 9 \geq k^2 + 2k + 4 = (k+1)^2 + 3$ since $k \geq 2$. Therefore $P(k+1)$ is true.

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We want to prove that $P(k+1)$ is true, i.e., $3^{(k+1)} \geq (k+1)^2 + 3 = k^2 + 2k + 4$.
Note that $3^{(k+1)} = 3(3^k) \geq 3(k^2 + 3)$ by the inductive hypothesis. From this we have $3(k^2 + 3) = 2k^2 + k^2 + 9 \geq k^2 + 2k + 4 = (k+1)^2 + 3$ since $k \geq 2$.
Therefore $P(k+1)$ is true.

⑤ **The result follows for all $n \geq 2$ by induction.**

Induction with a stricter hypothesis

Proving theorems by induction on a stricter hypothesis.

Changing the hypothesis

Can we use induction to prove $P(n)$ if we can't show that $P(k) \to P(k+1)$ in the inductive step?

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Find a predicate $Q(n)$ such that

$Q(n) \to P(n)$, $Q(0)$, and $Q(k) \to Q(k+1)$.

Changing the hypothesis

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Usually, but it requires a creative leap!

Find a predicate $Q(n)$ such that

$Q(n) \to P(n)$, $Q(0)$, and $Q(k) \to Q(k+1)$.

Then, we can prove $Q(n)$ by induction and get $P(n)$ by MP.

We call this using a stricter (or stronger) hypothesis.

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We will try to show that $P(n)$ is true for every integer $n \geq 1$ by induction.

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③ **Inductive hypothesis:**

Suppose that $P(k)$ is true for an arbitrary integer $k \geq 1$.

④ **Inductive step:**

We want to prove that $P(k+1)$ is true, i.e, $\sum_{i=1}^{k+1} 1/i^2 < 2$. Note that $\sum_{i=1}^{k+1} 1/i^2 = \sum_{i=1}^{k} 1/i^2 + 1/(k+1)^2$. By the inductive hypothesis, we know that $\sum_{i=1}^{k} 1/i^2 < 2$, and from this we can conclude that $\sum_{i=1}^{k} 1/i^2 + 1/(k+1)^2 < 2 + 1/(k+1)^2$. But this is not $P(k+1)$!

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We want to prove that $P(k+1)$ is true, i.e, $\sum_{i=1}^{k+1} 1/i^2 < 2$. Note that $\sum_{i=1}^{k+1} 1/i^2 = \sum_{i=1}^{k} 1/i^2 + 1/(k+1)^2$. By the inductive hypothesis, we know that $\sum_{i=1}^{k} 1/i^2 < 2$, and from this we can conclude that $\sum_{i=1}^{k} 1/i^2 + 1/(k+1)^2 < 2 + 1/(k+1)^2$. But this is not $P(k+1)$!

We need a stricter hypothesis! Let's try $Q(n) := \sum_{i=1}^{n} 1/i^2 \leq 2 - 1/n$, noting that $Q(n) \to P(n)$ for all $n >= 1$.

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We will show that $Q(n)$ is true for every integer $n \geq 1$ by induction.

② **Base case ($n = 1$):**
$\sum_{i=1}^{1} 1/i^2 = 1/1^2 = 1 \leq 2 - 1/1 = 1$ so $Q(1)$ is true.

Prove instead $\sum_{i=1}^{n} 1/i^2 \leq 2 - 1/n$ for all $n \geq 1$

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$\sum_{i=1}^{1} 1/i^2 = 1/1^2 = 1 \leq 2 - 1/1 = 1$ so $Q(1)$ is true.

③ **Inductive hypothesis:**
Suppose that $Q(k)$ is true for an arbitrary integer $k \geq 1$.

Prove instead $\sum_{i=1}^{n} 1/i^2 \leq 2 - 1/n$ for all $n \geq 1$

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$\sum_{i=1}^{1} 1/i^2 = 1/1^2 = 1 \leq 2 - 1/1 = 1$ so $Q(1)$ is true.

③ **Inductive hypothesis:**

Suppose that $Q(k)$ is true for an arbitrary integer $k \geq 1$.

④ **Inductive step:**

We want to prove that $Q(k+1)$ is true, i.e, $\sum_{i=1}^{k+1} 1/i^2 \leq 2 - 1/(k+1)$. Note that $\sum_{i=1}^{k+1} 1/i^2 = \sum_{i=1}^{k} 1/i^2 + 1/(k+1)^2$. By the inductive hypothesis, we know that $\sum_{i=1}^{k} 1/i^2 \leq 2 - 1/k$. Adding $1/(k+1)^2$ to both sides, we get $\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{(k+1)^2 - k}{k(k+1)^2} = 2 - \frac{k^2+k+1}{k(k+1)^2} = 2 - \frac{k(k+1)+1}{k(k+1)^2}$. Note that $\frac{k(k+1)}{k(k+1)^2} < \frac{k(k+1)+1}{k(k+1)^2}$ when $k \geq 1$, so $\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{k(k+1)+1}{k(k+1)^2} < 2 - \frac{k(k+1)}{k(k+1)^2}$ $= 2 - \frac{k+1}{(k+1)^2} = 2 - \frac{1}{k+1}$. This shows that $Q(k+1)$ holds.

Prove instead $\sum_{i=1}^{n} 1/i^2 \leq 2 - 1/n$ for all $n \geq 1$

① **Let $Q(n)$ be $\sum_{i=1}^{n} 1/i^2 \leq 2 - 1/n$.**

We will show that $Q(n)$ is true for every integer $n \geq 1$ by induction.

② **Base case ($n = 1$):**

$\sum_{i=1}^{1} 1/i^2 = 1/1^2 = 1 \leq 2 - 1/1 = 1$ so $Q(1)$ is true.

③ **Inductive hypothesis:**

Suppose that $Q(k)$ is true for an arbitrary integer $k \geq 1$.

④ **Inductive step:**

We want to prove that $Q(k+1)$ is true, i.e, $\sum_{i=1}^{k+1} 1/i^2 \leq 2 - 1/(k+1)$. Note that $\sum_{i=1}^{k+1} 1/i^2 = \sum_{i=1}^{k} 1/i^2 + 1/(k+1)^2$. By the inductive hypothesis, we know that $\sum_{i=1}^{k} 1/i^2 \leq 2 - 1/k$. Adding $1/(k+1)^2$ to both sides, we get $\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{(k+1)^2 - k}{k(k+1)^2} = 2 - \frac{k^2+k+1}{k(k+1)^2} = 2 - \frac{k(k+1)+1}{k(k+1)^2}$. Note that $\frac{k(k+1)}{k(k+1)^2} < \frac{k(k+1)+1}{k(k+1)^2}$ when $k \geq 1$, so $\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{k(k+1)+1}{k(k+1)^2} < 2 - \frac{k(k+1)}{k(k+1)^2}$ $= 2 - \frac{k+1}{(k+1)^2} = 2 - \frac{1}{k+1}$. This shows that $Q(k+1)$ holds.

⑤ **The result follows for all $n \geq 1$ by induction.**

Summary

Induction lets us prove statements about all natural numbers.

A proof by induction must show that $P(0)$ is true (*base case*).

And it must use the *inductive hypothesis* $P(k)$ to show that $P(k+1)$ is true (*inductive step*).

Induction also lets us prove theorems about integers $n \geq b$ for $b \in \mathbb{Z}$.

Adjust all parts of the proof to use $n \geq b$ instead of $n \geq 0$.

Induction proofs can use a hypothesis that implies $P(k)$.

It is not always possible to use $P(k)$ as the induction hypothesis.

In those cases, we use induction to prove $Q(n)$, where $Q(n) \rightarrow P(n)$.