# CSE 311 Lecture 11: Modular Arithmetic

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## Topics

**Sets and set operations**

A quick wrap-up of Lecture 10.

**Modular arithmetic basics**

Arithmetic over a finite domain (a.k.a computer arithmetic).

**Modular arithmetic properties**

Congruence, addition, multiplication, proofs.

**Modular arithmetic and integer representations**

Unsigned, sign-magnitude, and two's complement representation.

**Applications of modular arithmetic**

Hashing, pseudo-random numbers, ciphers.

## Sets and set operations

A quick wrap-up of Lecture 10.

## Sets and set operations

**A set is a collection of *elements*.**

Write $a \in B$ (or $a \notin B$) to say that $a$ is (or isn't) an element in the set $B$.

The order of elements doesn't matter, and duplicates don't matter.

**Sets $A$ and $B$ are *equal* if they have the same elements.**

$$A = B \equiv \forall x.\, x \in A \leftrightarrow x \in B$$

**$A$ is a *subset* of $B$ if every element of $A$ is also in $B$.**

$$A \subseteq B \equiv \forall x.\, x \in A \rightarrow x \in B$$

**Sets can be built from predicates or using set operations.**

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$
$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$
$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\}$$
$$A \oplus B = \{x : (x \in A) \oplus (x \in B)\}$$
$$\overline{A} = \{x : x \notin A\} = \{x : \neg(x \in A)\}$$

## This is Boolean algebra again

**Union $\cup$ is defined using $\vee$.**

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

**Intersection $\cap$ is defined using $\wedge$.**

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

**Complement works like $\neg$.**

$$\overline{A} = \{x : x \notin A\} = \{x : \neg(x \in A)\}$$

This means that all equivalences from Boolean algebra translate directly into set theory, and you can use them in your proofs!

## More sets

Power set, Cartesian product, and Russell’s paradox.

## Power set

Power set of a set $A$ is the set of all subsets of $A$.

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Examples

Let $\text{Days} = \{M, W, F\}$.

$\mathcal{P}(\text{Days}) =$

$\mathcal{P}(\varnothing) =$

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**Examples**

Let $\text{Days} = \{M, W, F\}$.

$\mathcal{P}(\text{Days}) = \{\varnothing, \{M\}, \{W\}, \{F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M, W, F\}\}$

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$\mathcal{P}(\varnothing) = \{\varnothing\} \neq \varnothing$

## Cartesian product

**The Cartesian product of two sets is the set of all of their ordered pairs.**

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

**Examples**

$\mathbb{R} \times \mathbb{R}$ is the real plane.

$\mathbb{Z} \times \mathbb{Z}$ is the set of all pairs of integers.

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$A \times \varnothing = \{(a, b) : a \in A \wedge b \in \varnothing\} = \{(a, b) : a \in A \wedge \mathsf{F}\} = \varnothing$.

## Russell’s paradox

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**To avoid the paradox …**

Define $S$ with respect to a universe of discourse.

$$S = \{x \in U : x \notin x\}$$

With this definition, $S \notin S$ and there is no contradiction because $S \notin U$.

## Working with sets

Representing sets as bitvectors and applications of bitvectors.

## Representing sets as bitvectors

Suppose that universe $U$ is $\{1, 2, \dots, n\}$.

We can represent every set $B \subseteq U$ as a vector of bits:

$$\begin{aligned} b_1 b_2 \dots b_n \text{ where } b_i &= 1 \text{ if } i \in B \\ b_i &= 0 \text{ if } i \notin B \end{aligned}$$

This is called the *characteristic vector* of set $B$.

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**Given characteristic vectors for $A$ and $B$, what is the vector for**

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$A \cup B = (a_1 + b_1) \ldots (a_n + b_n)$

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$$A \cup B = (a_1 + b_1) \ldots (a_n + b_n)$$
$$A \cap B = (a_1 \cdot b_1) \ldots (a_n \cdot b_n)$$

## Unix/Linux file permissions

```
$ ls -l
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

**Permissions maintained as bitvectors.**

Letter means the bit is 1.

”-“ means the bit is zero.

## Bitwise operations

$$\begin{array}{r} 01101101 \\ \vee\ 00110111 \\ \hline 01111111 \end{array}$$

```
z = x | y
```

$$\begin{array}{r} 00101010 \\ \wedge\ 00001111 \\ \hline 00001010 \end{array}$$

```
z = x & y
```

$$\begin{array}{r} 01101101 \\ \oplus\ 00110111 \\ \hline 01011010 \end{array}$$

```
z = x ^ y
```

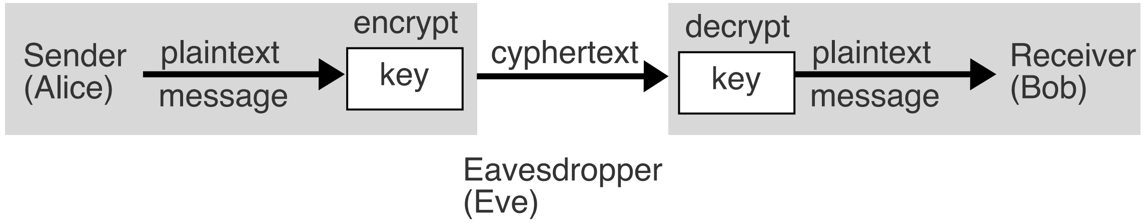
Note that $(x \oplus y) \oplus y = x$.

## Private key cryptography

Alice wants to communicate a message $m$ secretly to Bob, so that eavesdropper Eve who sees their conversation can't understand $m$.

Alice and Bob can get together ahead of time and privately share a secret key $K$.

How can they communicate securely in this setting?



## One-time pad

**Alice and Bob privately share a random $n$-bitvector $K$.**

Eve doesn't know $K$.

**Later, Alice has $n$-bit message $m$ to send to Bob.**

Alice computes $C = m \oplus K$.

Alice sends $C$ to Bob.

Bob computes $m = C \oplus K$, which is $(m \oplus K) \oplus K = m$.

**Eve can't figure out $m$ from $C$ unless she can guess $K$.**

And that's very unlikely for large $n$ …

## Modular arithmetic basics

Arithmetic over a finite domain (a.k.a computer arithmetic).

## Modular arithmetic in action

What does this Java program output?

```
public class Seconds {
   final static int SEC_IN_YEAR = 364*24*60*60;
   public static void main(String args[]) {
       System.out.println("Number of seconds in 10100 years: " +
                          (SEC_IN_YEAR*10100));
   }
}
```

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}
```

```
$javac Test.java
$java -Xmx128M -Xms16M Test
Number of seconds in 10100 years: -186619904
```

You'll recognize this as "integer overflow." It happens because computers use *modular arithmetic* to operate on finite integer data types, such as `int`.

## It all starts with divisibility …

*Divisibility* is the core concept behind modular arithmetic.

**Definition:** $a$ **divides** $b$**, written as** $a|b$**.**

For $a \in \mathbb{Z}, b \in \mathbb{Z}, a|b \leftrightarrow \exists k \in \mathbb{Z}.\, b = ka$.

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Examples: which of the following are true and which are false?

| | | | |
|---|---|---|---|
| $5\|1$ | $25\|5$ | $5\|0$ | $2\|3$ |
| $1\|5$ | $5\|25$ | $0\|5$ | $3\|2$ |

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## Division theorem

**Division theorem**

For $a \in \mathbb{Z}, d \in \mathbb{Z}$ with $d > 0$,
there exist *unique* integers $q, r$ with $0 \leq r < d$
such that $a = dq + r$.

If we divide $a$ by $d$, we get a unique quotient $q = a \operatorname{div} d$ and non-negative remainder $r = a \bmod d$.

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Note that $r \geq 0$ even if $a < 0$, so mod is not `%`.

```java
public class NotMod {
   public static void main(String args[]) {
       System.out.println("-5 mod 2 = 1.");
       System.out.println("-5 % 2 = " + (-5 % 2));
   }
}
```

```
$javac NotMod.java
$java -Xmx128M -Xms16M NotMod
-5 mod 2 = 1.
-5 % 2 = -1
```

## Example: arithmetic mod 7

$$a +_7 b = (a + b) \bmod 7$$

| + | **0** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| **2** | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| **3** | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| **4** | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| **5** | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| **6** | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

$$a \times_7 b = (a \times b) \bmod 7$$

| * | **0** | **1** | **2** | **3** | **4** | **5** | **6** |
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| **6** | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

## Modular arithmetic properties

Congruence, addition, multiplication, proofs.

## Congruence modulo a positive integer

**Definition:** $a$ **is congruent to** $b$ **modulo** $m$**, written as** $a \equiv b \pmod{m}$

For $a, b, m \in \mathbb{Z}$ with $m > 0$, $a \equiv b \pmod{m} \leftrightarrow m|(a - b)$

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**Examples: what do these mean and when are they true?**

$x \equiv 0 \pmod{2}$

$-1 \equiv 19 \pmod{5}$

$y \equiv 2 \pmod{7}$

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True for every $x$ that is divisible by 2, i.e., even.

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True because $-1 - 19 = -20$ is divisible by 5.

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True because $-1 - 19 = -20$ is divisible by 5.

$y \equiv 2 \pmod{7}$

True for every $y$ of the form $y = 2 + 7k$ where $k \in \mathbb{Z}$.

## Congruence and equality

**Congruence property**

Let $a, b, m \in \mathbb{Z}$ with $m > 0$.

Then, $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

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**Proof:**

Suppose that $a \equiv b \pmod{m}$.

Suppose that $a \bmod m = b \bmod m$.

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**Proof:**

Suppose that $a \equiv b \pmod{m}$.

Then $m | a - b$ by definition of congruence. So $a - b = km$ for some $k \in \mathbb{Z}$ by definition of divides. Therefore, $a = b + km$. Taking both sides modulo $m$, we get $a \bmod m = (b + km) \bmod m = b \bmod m$.

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Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and $b = ms + (b \bmod m)$ for some $q, s \in \mathbb{Z}$. Then, $a - b = (mq + (a \bmod m)) - (ms + (b \bmod m)) = m(q - s) + (a \bmod m - b \bmod m) = m(q - s)$, since $a \bmod m = b \bmod m$. Therefore, $m | (a - b)$ and so $a \equiv b \pmod{m}$.

## The $\bmod m$ function vs the $\equiv \pmod{m}$ predicate

The $\bmod m$ function takes any $a \in \mathbb{Z}$ and maps it to a remainder $a \bmod m \in \{0, 1, \dots, m-1\}$.

In other words, $\bmod m$ places all integers that have the same remainder modulo $m$ into the same “group” (a.k.a. “congruence class”).

The $\equiv \pmod{m}$ predicate compares $a, b \in \mathbb{Z}$ and returns true if and only if $a$ and $b$ are in the same group according to the $\bmod m$ function.

## Modular addition property

**Modular addition property**

Let $m$ be a positive integer ($m \in \mathbb{Z}$ with $m > 0$).

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$.

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**Proof:**

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. By the definition of congruence, there are $k$ and $j$ such that $a - b = km$ and $c - d = jm$. Adding these equations together, we get $(a + c) - (b + d) = m(j + k)$. Reapplying the definition of congruence, we get that $(a + c) \equiv (b + d) \pmod{m}$.

## Modular multiplication property

**Modular multiplication property**

Let $m$ be a positive integer ($m \in \mathbb{Z}$ with $m > 0$).
If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.

# Modular multiplication property

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Let $m$ be a positive integer ($m \in \mathbb{Z}$ with $m > 0$).
If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.

**Proof:**

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. By the definition of congruence, there are $k$ and $j$ such that $a - b = km$ and $c - d = jm$. So, $a = km + b$ and $c = jm + b$. Multiplying these equations together, we get $ac = (km + b)(jm + d) = kjm^2 + kmd + bjm + bd$. Rearranging gives us $ac - bd = m(kjm + kd + bj)$. Reapplying the definition of congruence, we get that $ac \equiv bd \pmod{m}$.

## Example: a proof using modular arithmetic

Let $n \in \mathbb{Z}$, and prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$.

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**Let's look at a few examples:**

$$\begin{aligned} 0^2 &= 0 \equiv 0 \pmod{4} \\ 1^2 &= 1 \equiv 1 \pmod{4} \\ 2^2 &= 4 \equiv 0 \pmod{4} \\ 3^2 &= 9 \equiv 1 \pmod{4} \\ 4^2 &= 16 \equiv 0 \pmod{4} \end{aligned}$$

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4^2 &= 16 \equiv 0 \pmod 4
\end{aligned}$$

**It looks like**

$$\begin{aligned}
n \equiv 0 \pmod 2 &\to n^2 \equiv 0 \pmod 4\\
n \equiv 1 \pmod 2 &\to n^2 \equiv 1 \pmod 4
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## Example: a proof using modular arithmetic

Let $n \in \mathbb{Z}$, and prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$.

**Let's look at a few examples:**

$$\begin{aligned} 0^2 &= 0 \equiv 0 \pmod 4 \\ 1^2 &= 1 \equiv 1 \pmod 4 \\ 2^2 &= 4 \equiv 0 \pmod 4 \\ 3^2 &= 9 \equiv 1 \pmod 4 \\ 4^2 &= 16 \equiv 0 \pmod 4 \end{aligned}$$

**It looks like**

$$n \equiv 0 \pmod 2 \rightarrow n^2 \equiv 0 \pmod 4$$
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**Proof by cases:**

Case 1 ($n$ is even):

Case 2 ($n$ is odd):

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Case 2 ($n$ is odd):

Suppose $n \equiv 1 \pmod{2}$. Then $n = 2k + 1$ for some integer $k$. So $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Therefore, by definition of congruence, $n^2 \equiv 1 \pmod{4}$.

## Modular arithmetic and integer representations

Unsigned, sign-magnitude, and two’s complement representation.

## Unsigned integer representation

**Represent integer $x$ as a sum of $n$ powers of 2:**

If $x = \sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0, 1\}$,
then the representation is $b_{n-1} \dots b_2 b_1 b_0$.

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$99 = 64 + 32 + 2 + 1$
$18 = 16 + 2$

**So for $n = 8$:**

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This works for unsigned integers. How do we represented signed integers?

## Sign-magnitude integer representation

**If $-2^{n-1} < x < 2^{n-1}$, represent $x$ with $n$ bits as follows:**

Use the first bit as the sign (0 for positive and 1 for negative), and the remaining $n - 1$ bits as the (unsigned) value.

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The problem with this representation is that our standard arithmetic algorithms no longer work, e.g., adding the representation of -18 and 99 doesn't give the representation of 81.

## Two’s complement integer representation

**Represent $x$ with $n$ bits as follows:**

If $0 \leq x < 2^{n-1}$, use the $n$-bit unsigned representation of $x$.

If $-2^{n-1} \leq x < 0$, use the $n$-bit unsigned representation of $2^n - |x|$.

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**Key property:**

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**Key property:**

Two’s complement representation of any number $y$ is equivalent to $y \bmod 2^n$ so arithmetic works $\bmod 2^n$.

**Examples:**

$99 = 64 + 32 + 2 + 1$
$18 = 16 + 2$
$2^8 - 18 = 256 - 18 = 238 = 128 + 64 + 32 + 8 + 4 + 2$
$81 = 64 + 16 + 1$

**So for $n = 8$:**

$99 = 0110\ 0011$
$-18 = 1110\ 1110$
$81 = 0101\ 0001$

## Computing the two's complement representation

For $-2^{n-1} \leq x < 0, x$ is represented using the $n$-bit unsigned representation of $2^n - |x|$.

Here is an easy way to compute this value:

- Compute the $n$-bit unsigned representation of $|x|$.
- Flip the bits of $|x|$ to get the representation of $2^n - 1 - |x|$.
  - This works because the string of all 1's represents $2^n - 1$.
- Add 1 to get $2^n - |x|$.

## Applications of modular arithmetic

Hashing, pseudo-random numbers, ciphers.

## Hashing

**Problem:**

We want to map a small number of data values from a large domain $\{0, 1, \dots, M-1\}$ into a small set of locations $\{0, 1, \dots, n-1\}$ to be able to quickly check if a value is present.

**Solution:**

Compute $\mathrm{hash}(x) = x \bmod p$ for a prime $p$ close to $n$.

Or, compute $\mathrm{hash}(x) = ax + b \bmod p$ for a prime $p$ close to $n$.

**This approach depends on all of the bits of data the data.**

Helps avoid collisions due to similar values.

But need to manage them if they occur.

## Pseudo-random number generation

**Linear Congruential method**

$$x_{n+1} = (ax_n + c) \bmod m$$

Choose random $x_0$, $a$, $c$, $m$ and produce a long sequences of $x_n$'s.

## Simple ciphers

**Ceasar or shift cipher:**

Treat letters as numbers: A = 0, B = 1, …

$$f(p) = (p + k) \bmod 26$$
$$f^{-1}(p) = (p - k) \bmod 26$$

**More general version:**

$$f(p) = (ap + b) \bmod 26$$
$$f^{-1}(p) = (a^{-1}(p - b)) \bmod 26$$

## Summary

**Sets can be represented efficiently using bitvectors.**

This representation is used heavily in the real world.

With this representation, set operations reduce to fast bitwise operations.

**Modular arithmetic is arithmetic over a finite domain.**

Key notions are divisibility and congruency $\bmod m$.

**Modular arithmetic is the basis of computing.**

Used with two's complement representation to implement computer arithmetic.

Also used in hashing, pseudo-random number generation, and cryptography.