CSE 311 Lecture 10: Set Theory

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Topics

English proofs and proof strategies

A quick wrap-up of Lecture 09.

Set theory basics

Set membership ($\in$), subset ($\subseteq$), and equality ($=$).

Set operations

Set operations and their relation to Boolean algebra.

More sets

Power set, Cartesian product, and Russell's paradox.

Working with sets

Representing sets as bitvectors and applications of bitvectors.

English proofs and proof strategies

A quick wrap-up of Lecture 09.

Benefits of English proofs

This is more work to write

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%a = add %i, 1
%b = mod %a, %n
%c = add %arr, %b
%d = load %c
%e = add %arr, %i
store %e, %d
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than this

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arr[i] = arr[(i+1) % n];
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Higher level language is easier because it skips details.

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Formal proofs are the low level language: each part must be spelled out in precise detail.

English proofs are the high level language.

An English proof is correct if the *reader* is convinced they can “compile” it to a formal proof if necessary.

Proof strategies

Sometimes, it's too hard to prove a theorem directly using inference rules, equivalences, and domain properties.

When that's the case, try one of the following alternative strategies:

- Proof by contrapositive,
- Disproof by counterexamples, and
- Proof by contradiction.

Proof by contrapositive

If we assume $\neg q$ and derive $\neg p$, then we have proven that $\neg q \to \neg p$, which is equivalent to proving $p \to q$.

1.1. $\neg q$ **Assumption**

$\ldots$

1.3. $\neg p$

2. $\neg q \to \neg p$ **Direct Proof Rule**
3. $p \to q$ **Contrapositive: 2**

Counterexamples

To *disprove* $\forall x.\ P(x)$, prove $\exists x.\ \neg P(x)$.

Works by DeMorgan's Law: $\neg\forall x.\ P(x) \equiv \exists x.\ \neg P(x)$.

All we need to do is find an x for which $P(x)$ is false.

This x is called a *counterexample*.

Example: disprove that "Every prime number is odd".

2 is a prime number that is not odd.

Proof by contradiction

If we assume p and derive F (a contradiction), then we have proven $\neg p$.

1.1.	p	**Assumption**
	$\dots$	
1.3.	F	
2.	$p \to \mathsf{F}$	**Direct Proof Rule**
3.	$\neg p \vee \mathsf{F}$	**Law of Implication: 2**
4.	$\neg p$	**Identity: 3**

An example proof by contradiction

Prove that "No integer is both even and odd."

English proof: $\neg\exists x.\ \mathrm{Even}(x) \land \mathrm{Odd}(x) \equiv \forall x.\ \neg(\mathrm{Even}(x) \land \mathrm{Odd}(x))$.

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Let x be an arbitrary integer and suppose that it is both even and odd.

Then $x = 2a$ for some integer a and $x = 2b + 1$ for some integer b.

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Therefore $2a = 2b + 1$ and hence $a = b + \frac{1}{2}$.

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But two integers cannot differ by $\frac{1}{2}$ so this is a contradiction.
Therefore no integer is both even and odd. $\square$

Fun strategy: proof by computer

Use an automated theorem prover:

```
; No integer is both even and odd.

(define-fun even ((x Int)) Bool
  (exists ((y Int)) (= x (* 2 y))))

(define-fun odd ((x Int)) Bool
  (exists ((y Int)) (= x (+ (* 2 y) 1))))

(define-fun claim () Bool
  (not (exists ((x Int)) (and (even x) (odd x)))))

(assert (not claim)) ; proof by contradiction

(check-sat)
```

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While this example works, proofs of arbitrary formulas in predicate logic *cannot* be automated. But *interactive theorem provers* can still help by checking your formal proof and filling in some low-level details for you.

Fun fact: counterexamples & contradiction in verification

Automated verifiers work by counterexample and contradiction proofs.

Recall that program verification involves proving that a program P satisfies a specification S on all inputs x: $\forall x.\, p(x) \rightarrow s(x)$, where p and s are formulas encoding the semantics of P and S.

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The program verifier sends the formula $\exists x.\, p(x) \wedge \neg s(x)$ to the prover.

$\neg\forall x.\, p(x) \rightarrow s(x) \equiv \exists x.\, \neg(p(x) \rightarrow s(x)) \equiv \exists x.\, \neg(\neg p(x) \vee s(x)) \equiv \exists x.\, p(x) \wedge \neg s(x)$.

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If the prover finds a counterexample, we know the program is incorrect.

The counterexample is a concrete input (test case) on which the program violates the spec.

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If the prover finds a counterexample, we know the program is incorrect.

The counterexample is a concrete input (test case) on which the program violates the spec.

If no counterexample exists, we know the program is correct.

Because this is proof by contradiction! The prover assumed $\exists x.\, p(x) \wedge \neg s(x)$ and arrived at false (“unsat”).

Set theory basics

Set membership ($\in$), subset ($\subseteq$), and equality ($=$).

What is a set?

A set is a collection of objects called *elements*.

Write $a \in B$ to say that a is an element in the set B.

Write $a \notin B$ to say that a isn't an element of B.

Examples

$A = \{1\}$

$B = \{1, 3, 2\}$

$C = \{\triangle, 1\}$

$D = \{\{17\}, 17\}$

$E = \{1, 2, 7, \alpha, \emptyset, \text{dog},$  $\}$

Some common sets

$\mathbb{N}$ is the set of **Natural Numbers**: $\mathbb{N} = \{0, 1, 2, \ldots\}$
$\mathbb{Z}$ is the set of **Integers**: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$ is the set of **Rational Numbers**: e.g. $\frac{1}{2}, -17, \frac{32}{48}$
$\mathbb{R}$ is the set of **Real Numbers**: e.g. $1, -17, \frac{32}{48}, \pi, \sqrt{2}$
$[n]$ is the set $\{1, 2, \ldots, n\}$ where n is a natural number.
$\{\} = \varnothing$ is the **empty set**; the *only* set with no elements.

Sets can be elements of other sets

For example, consider the sets

$$A = \{\{1\}, \{2\}, \{1, 2\}, \varnothing\}$$
$$B = \{1, 2\}$$

Then we have

$$B \in A$$
$$\varnothing \in A$$

Definitions: equality and subset

A and B are *equal* if they have the same elements.

$$A = B \equiv \forall x.\, x \in A \leftrightarrow x \in B$$

A is a *subset* of B if every element of A is also in B.

$$A \subseteq B \equiv \forall x.\, x \in A \rightarrow x \in B$$

Note: $A = B \equiv A \subseteq B \wedge B \subseteq A$.

Example: understanding equality

A and B are *equal* if they have the same elements.

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Which sets are equal to each other?

$$A = \{1, 2, 3\}$$
$$B = \{3, 4, 5\}$$
$$C = \{3, 4\}$$
$$D = \{4, 3, 3\}$$
$$E = \{3, 4, 3\}$$
$$F = \{4, \{3\}\}$$

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$F = \{4, \{3\}\}$

C, D, E are equal to each other.

Order of elements doesn't matter, and duplicates don't matter.

F is *not* equal to C, D, E because $\{3\} \neq 3$!

Example: understanding subsets

A is a *subset* of B if every element of A is also in B.

$$A \subseteq B \equiv \forall x.\, x \in A \to x \in B$$

Example sets

$$A = \{1, 2, 3\}$$
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Are these subset formulas true or false?

$$\emptyset \subseteq A$$
$$A \subseteq B$$
$$C \subseteq B$$

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Building sets from predicates

$$S = \{x : P(x)\}$$

S is the set of all x in the domain of P for which $P(x)$ is true.
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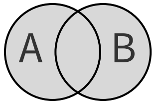
Set operations

Set operations and their relation to Boolean algebra.

Union, intersection, and set difference

Union

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$



Intersection

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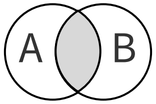
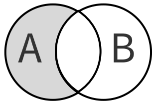


Set difference

$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\}$$



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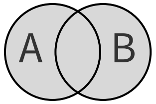

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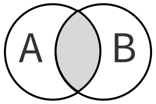

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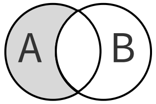

Given the following sets …

$$A = \{1, 2, 3\}$$
$$B = \{3, 5, 6\}$$
$$C = \{3, 4\}$$

Use set operations to make:

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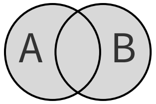

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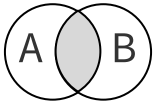

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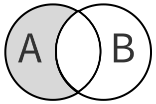

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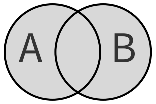

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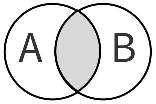

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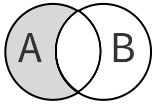

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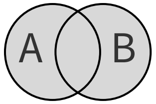

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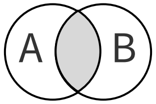

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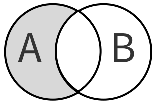

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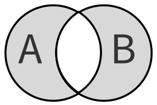
Symmetric difference and complement

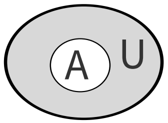
Symmetric difference

$$A \oplus B = \{x : (x \in A) \oplus (x \in B)\}$$



Complement (with respect to universe U)

$$\overline{A} = \{x : x \notin A\} = \{x : \neg(x \in A)\}$$



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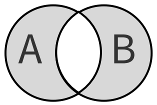

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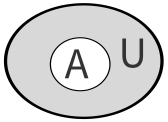

Given the sets and universe …

$$A = \{1, 2, 3\}$$
$$B = \{1, 2, 4, 6\}$$
$$U = \{1, 2, 3, 4, 5, 6\}$$

What is

$$A \oplus B =$$
$$\overline{A} =$$

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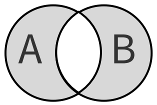

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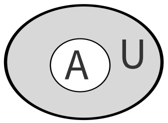

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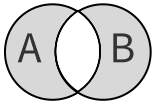

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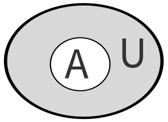

Given the sets and universe …

$$A = \{1, 2, 3\}$$
$$B = \{1, 2, 4, 6\}$$
$$U = \{1, 2, 3, 4, 5, 6\}$$

What is

$$A \oplus B = \{3, 4, 6\}$$
$$\overline{A} = \{4, 5, 6\}$$

This is Boolean algebra again

Union $\cup$ is defined using $\vee$.

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

Intersection $\cap$ is defined using $\wedge$.

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

Complement works like $\neg$.

$$\overline{A} = \{x : x \notin A\} = \{x : \neg(x \in A)\}$$

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This means that all equivalences from Boolean algebra translate directly into set theory, and you can use them in your proofs!

DeMorgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

How would we prove these?

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To prove $C = D$, show
$x \in C \rightarrow x \in D$ and
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Then, by definition of complement, $\neg(x \in A \cup B)$.

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Applying DeMorgan's laws, we get $x \notin A \land x \notin B$.
So, $x \in \overline{A} \land x \in \overline{B}$ by definition of complement.
Finally, $x \in \overline{A} \cap \overline{B}$ by definition of "$\cap$", and we have shown that $x \in \overline{A \cup B} \rightarrow x \in \overline{A} \cap \overline{B}$.

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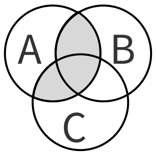
Next, let $x \in \overline{A} \cap \overline{B}$ be arbitrary.
…
Finally, $x \in \overline{A \cup B}$, so $x \in \overline{A} \cap \overline{B} \rightarrow x \in \overline{A \cup B}$, which completes the proof. □

Distributivity laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



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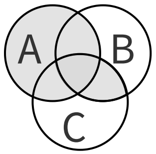

A simple set proof

Prove that for any sets A and B, we have $(A \cap B) \subseteq A$.

Recall that $X \subseteq Y \equiv \forall x.\, x \in X \to x \in Y$

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It follows that $x \in A$, as required. $\square$

Set proofs can use Boolean algebra equivalences

Prove that for any sets A and B, we have $(A \cap B) \cup (A \cap \overline{B}) = A$.

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$$\begin{aligned} (A \cap B) \cup (A \cap \overline{B}) &= A \cap (B \cup \overline{B}) && \text{Distributivity} \\ &= A \cap U && \text{Complementarity} \\ &= A && \text{Identity} \end{aligned}$$

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Universe U corresponds to 1 and $\varnothing$ corresponds to 0.

More sets

Power set, Cartesian product, and Russell's paradox.

Power set

Power set of a set A is the set of all subsets of A.

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Examples

Let $\text{Days} = \{M, W, F\}$.

$\mathcal{P}(\text{Days}) =$

$\mathcal{P}(\varnothing) =$

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$\mathcal{P}(\varnothing) = \{\varnothing\} \neq \varnothing$

Cartesian product

The Cartesian product of two sets is the set of all of their ordered pairs.

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

Examples

$\mathbb{R} \times \mathbb{R}$ is the real plane.

$\mathbb{Z} \times \mathbb{Z}$ is the set of all pairs of integers.

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$A \times \varnothing = \{(a, b) : a \in A \wedge b \in \varnothing\} = \{(a, b) : a \in A \wedge \mathsf{F}\} = \varnothing$.

Russell's paradox

Let S be the set of all sets that don't contain themselves.

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To avoid the paradox …

Define S with respect to a universe of discourse.

$S = \{x \in U : x \notin x\}$

With this definition, $S \notin S$ and there is no contradiction because $S \notin U$.

Working with sets

Representing sets as bitvectors and applications of bitvectors.

Representing sets as bitvectors

Suppose that universe U is $\{1, 2, \ldots, n\}$.

We can represent every set $B \subseteq U$ as a vector of bits:

$$b_1 b_2 \ldots b_n \text{ where } b_i = 1 \text{ if } i \in B$$
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This is called the *characteristic vector* of set B.

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$$A \cap B = (a_1 \wedge b_1) \ldots (a_n \wedge b_n)$$

Unix/Linux file permissions

```
$ ls -l
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

Permissions maintained as bitvectors.

Letter means the bit is 1.

”-“ means the bit is zero.

Bitwise operations

$$\begin{array}{r} 01101101 \\ \vee\ 00110111 \\ \hline 01111111 \end{array}$$

```
z = x | y
```

$$\begin{array}{r} 00101010 \\ \wedge\ 00001111 \\ \hline 00001010 \end{array}$$

```
z = x & y
```

$$\begin{array}{r} 01101101 \\ \oplus\ 00110111 \\ \hline 01011010 \end{array}$$

```
z = x ^ y
```

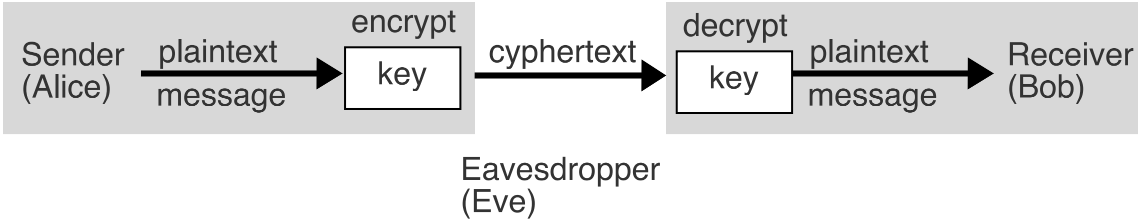
Note that $(x \oplus y) \oplus y = x$.

Private key cryptography

Alice wants to communicate a message m secretly to Bob, so that eavesdropper Eve who sees their conversation can't understand m.

Alice and Bob can get together ahead of time and privately share a secret key K.

How can they communicate securely in this setting?



One-time pad

Alice and Bob privately share a random n-bitvector K.

Eve doesn't know K.

Later, Alice has n-bit message m to send to Bob.

Alice computes $C = m \oplus K$.

Alice sends C to Bob.

Bob computes $m = C \oplus K$, which is $(m \oplus K) \oplus K = m$.

Eve can't figure out m from C unless she can guess K.

And that's very unlikely for large n …

Summary

Sets are a basic notion in mathematics and computer science.

Collections of objects called elements.

Can be compared for equality ($=$) and containment ($\subseteq$).

Set operations correspond to Boolean algebra operations.

You can prove theorems about sets using Boolean algebra laws.

Sets can be represented efficiently using bitvectors.

This representation is used heavily in the real world.

With this representation, set operations reduce to fast bitwise operations.