CSE 311 Lecture 02: Logic, Equivalence, and Circuits

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Topics

CSE 390Z

A workshop for CSE 311 students.

Propositional logic

A brief review of Lecture 01.

Classifying compound propositions

Converse, contrapositive, and inverse of implication.

Tautology, contradiction, contingency.

Logical equivalence

Equivalence, laws of logic, and properties of logical connectives.

Application: digital circuits

Gates, combinational circuits, and circuit equivalence.

CSE 390Z

A workshop for CSE 311 students.

Logistics of CSE 390Z

Build a community within 311, learn collaborative problem solving tactics, and practice study skills.

Meets Thursdays 3:30-4:50pm (maybe also 5:00-6:20pm).

For more information, email Nicole Riley at nriley16@uw.edu and check out the course website.

To sign up, request an add code.

Propositional logic

A brief review of Lecture 01.

Syntax and semantics of propositional logic

Syntax

Atomic propositions are “words” in propositional logic.

Propositional variables represent atomic propositions.

Compound propositions are “sentences” made with logical connectives:

$\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$.

Semantics

A variable is either true (T) or false (F).

Truth tables show the meaning of compound propositions.

Connectives and truth tables

p	$\neg p$
F	T
T	F

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Implication can be tricky but truth tables don't lie

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

- p implies q
- whenever p is true q must be true
- if p then q
- q if p
- p is sufficient for q
- p only if q
- q is necessary for p

In an implication $p \to q$:

- p is called the *premise* or *antecedent*.
- q is called the *conclusion* or *consequence*.

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English translations for $p \to q$ where p is "It's raining" and q is "I have my umbrella".

If it's raining, **then** I have my umbrella.

I have my umbrella **if** it's raining.

It's raining **only if** I have my umbrella.

Understanding biconditional (bi-implication)

Connective	Write as	Read as	True when
Biconditional	$p \leftrightarrow q$	"p if and only if q"	p, q have the same truth value

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

- p iff q
- p is equivalent to q
- p implies q and q implies p
- p is necessary and sufficient for q

Translating English sentences to logic

Garfield has black stripes ***if*** *he is an orange cat* ***and*** *likes lasagna,* ***and*** *he is an orange cat* ***or*** *does* ***not*** *like lasagna.*

p = "Garfield has black stripes."
q = "Garfield is an orange cat."
r = "Garfield likes lasagna."

↓ Step 1: abstract

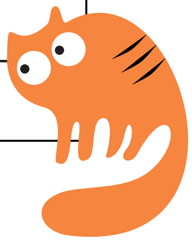
(p ***if*** *(q* ***and*** *r))* ***and*** *(q* ***or*** *(**not** r))*

↓ Step 2: replace English connectives with logical connectives

$$((q \wedge r) \rightarrow p) \wedge (q \vee (\neg r))$$

Understanding sentences with truth tables

p	q	r	$\neg r$	$(q \vee (\neg r))$	$(q \wedge r)$	$((q \wedge r) \rightarrow p)$	$((q \wedge r) \rightarrow p) \wedge (q \vee (\neg r))$
F	F	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	T	F	T	T	F	T	T
F	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
T	F	T	F	F	F	T	F
T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T



*Garfield has black stripes **if** he is an orange cat **and** likes lasagna, **and** he is an orange cat **or** does **not** like lasagna.*

p = "Garfield has black stripes."
q = "Garfield is an orange cat."
r = "Garfield likes lasagna."

Precedence of logical connectives

We'll use the following precedence rules for propositional connectives:

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

All operators are right-associative.

When in doubt, use parentheses.

Example: $\neg p \rightarrow q \vee r \leftrightarrow p \vee q \wedge r$

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$(\neg p) \rightarrow q \vee r \leftrightarrow p \vee q \wedge r$

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$(\neg p) \rightarrow q \vee r \leftrightarrow p \vee q \wedge r$

$(\neg p) \rightarrow q \vee r \leftrightarrow p \vee (q \wedge r)$

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$(\neg p) \rightarrow q \vee r \leftrightarrow p \vee q \wedge r$

$(\neg p) \rightarrow q \vee r \leftrightarrow p \vee (q \wedge r)$

$(\neg p) \rightarrow (q \vee r) \leftrightarrow (p \vee (q \wedge r))$

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$(\neg p) \rightarrow q \vee r \leftrightarrow p \vee q \wedge r$

$(\neg p) \rightarrow q \vee r \leftrightarrow p \vee (q \wedge r)$

$(\neg p) \rightarrow (q \vee r) \leftrightarrow (p \vee (q \wedge r))$

$((\neg p) \rightarrow (q \vee r)) \leftrightarrow (p \vee (q \wedge r))$

Classifying compound propositions

Converse, contrapositive, and inverse of implication.
Tautology, contradiction, contingency.

Implication and friends

Implication

$$p \to q$$

Converse

$$q \to p$$

Contrapositive

$$\neg q \to \neg p$$

Inverse

$$\neg p \to \neg q$$

How do these relate to each other?

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg q \to \neg p$	$\neg p \to \neg q$
F	F			T	T		
F	T			T	F		
T	F			F	T		
T	T			F	F		

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F	F	T	T	T	T		
F	T	T	F	T	F		
T	F	F	T	F	T		
T	T	T	T	F	F		

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How do these relate to each other?

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg q \to \neg p$	$\neg p \to \neg q$
F	F	T	T	T	T	T	
F	T	T	F	T	F	T	
T	F	F	T	F	T	F	
T	T	T	T	F	F	T	

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F	F	T	T	T	T	T	T
F	T	T	F	T	F	T	F
T	F	F	T	F	T	F	T
T	T	T	T	F	F	T	T

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F	F	T	T	T	T	T	T
F	T	T	F	T	F	T	F
T	F	F	T	F	T	F	T
T	T	T	T	F	F	T	T

An implication and its contrapositive have the same truth value!

Tautology, contradiction, and contingency

A compound proposition is a

- *Tautology* if it is always true;
- *Contradiction* if it is always false;
- *Contingency* if it can be either true or false.

$(p \to q) \land p$

$p \lor \neg p$

$p \oplus p$

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$(p \to q) \land p$

This is a contingency. It's true when $p = q = \mathsf{T}$ and false when $p = \mathsf{T}, q = \mathsf{F}$.

$p \lor \neg p$

$p \oplus p$

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This is a contingency. It's true when $p = q = \mathsf{T}$ and false when $p = \mathsf{T}, q = \mathsf{F}$.

$p \lor \neg p$

This is a tautology. It's true no matter what truth value p takes on.

$p \oplus p$

Tautology, contradiction, and contingency

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$(p \to q) \land p$
: This is a contingency. It's true when $p = q = \mathsf{T}$ and false when $p = \mathsf{T}, q = \mathsf{F}$.

$p \lor \neg p$
: This is a tautology. It's true no matter what truth value p takes on.

$p \oplus p$
: This is a contradiction. It's false no matter what truth value p takes on.

Logical equivalence

Equivalence, laws of logic, and properties of logical connectives.

Equivalence of compound propositions

*A and B are **logically equivalent**, written as $A \equiv B$, if they have the same truth values in all possible cases.*

$$p \wedge q \equiv p \wedge q$$

$$p \wedge q \equiv q \wedge p$$

$$p \wedge q \not\equiv q \vee p$$

Equivalence of compound propositions

> *A and B are* ***logically equivalent****, written as $A \equiv B$, if they have the same truth values in all possible cases.*

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Two formulas that are syntactically identical are also equivalent.

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Equivalence of compound propositions

*A and B are **logically equivalent**, written as $A \equiv B$, if they have the same truth values in all possible cases.*

$p \land q \equiv p \land q$
: Two formulas that are syntactically identical are also equivalent.

$p \land q \equiv q \land p$
: These two formulas are syntactically different but have the same truth table!

$p \land q \not\equiv q \lor p$

Equivalence of compound propositions

*A and B are **logically equivalent**, written as $A \equiv B$, if they have the same truth values in all possible cases.*

$p \land q \equiv p \land q$
Two formulas that are syntactically identical are also equivalent.

$p \land q \equiv q \land p$
These two formulas are syntactically different but have the same truth table!

$p \land q \not\equiv q \lor p$
When $p = \mathsf{T}$ and $q = \mathsf{F}$, $p \land q$ is false but $p \lor q$ is true!

$A \equiv B$ versus $A \leftrightarrow B$

$A \equiv B$ is an **assertion** that A and B have the same truth tables.

- This is *not* a compound proposition (sentence) in propositional logic!
- It is also sometimes called a *semantic judgement*.

$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of the variables that occur in A and B.

$A \equiv B$ versus $A \leftrightarrow B$

$A \equiv B$ is an **assertion** that A and B have the same truth tables.

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$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of the variables that occur in A and B.

$A \equiv B$ and $(A \leftrightarrow B) \equiv \top$ have the same meaning.

A and B are equivalent when $A \leftrightarrow B$ is a tautology.

Important equivalences: DeMorgan's laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

How do we check that an equivalence $A \equiv B$ holds?

Important equivalences: DeMorgan's laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

How do we check that an equivalence $A \equiv B$ holds?

Use truth tables to check that $A \leftrightarrow B$ is a tautology:

p	q	$\neg p$	$\neg q$	$\neg p \lor \neg q$	$p \land q$	$\neg(p \land q)$	$\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$
F	F	T	T				
F	T	T	F				
T	F	F	T				
T	T	F	F				

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F	F	T	T	T	F	T	T
F	T	T	F	T	F	T	T
T	F	F	T	T	F	T	T
T	T	F	F	F	T	F	T

Important equivalences: DeMorgan's laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

How do we check that an equivalence $A \equiv B$ holds?

Use truth tables to check that $A \leftrightarrow B$ is a tautology:

p	q	$\neg p$	$\neg q$	$\neg p \lor \neg q$	$p \land q$	$\neg(p \land q)$	$\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$
F	F	T	T	T	F	T	T
F	T	T	F	T	F	T	T
T	F	F	T	T	F	T	T
T	T	F	F	F	T	F	T

Fun fact: you can also use a theorem prover to check that $\neg(A \leftrightarrow B)$ is a contradiction!

Important equivalences: law of implication

$$p \to q \equiv \neg p \lor q$$

p	q	$p \to q$	$\neg p$	$\neg p \lor q$	$(p \to q) \leftrightarrow (\neg p \lor q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

Important equivalences: law of implication

$$p \to q \equiv \neg p \vee q$$

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

More equivalences related to implication

$$p \to q \equiv \neg q \to \neg p$$
$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

Important equivalences: properties of connectives

Identity

$$p \land \mathsf{T} \equiv p$$
$$p \lor \mathsf{F} \equiv p$$

Domination

$$p \land \mathsf{F} \equiv \mathsf{F}$$
$$p \lor \mathsf{T} \equiv \mathsf{T}$$

Idempotence

$$p \land p \equiv p$$
$$p \lor p \equiv p$$

Commutativity

$$p \land q \equiv q \land p$$
$$p \lor q \equiv q \lor p$$

Associativity

$$(p \land q) \land r \equiv p \land (q \land r)$$
$$(p \lor q) \lor r \equiv p \lor (q \lor r)$$

Distributivity

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$
$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

Absorption

$$p \land (p \lor q) \equiv p$$
$$p \lor (p \land q) \equiv p$$

Negation

$$p \land \neg p \equiv \mathsf{F}$$
$$p \lor \neg p \equiv \mathsf{T}$$

Double negation

$$p \equiv \neg\neg p$$

Application: digital circuits

Gates, combinational circuits, and circuit equivalence.

Computing with logic

Digital circuits implement propositional logic:

- T corresponds to 1 or high voltage.
- F corresponds to 0 or low voltage.

Digital gates are functions that

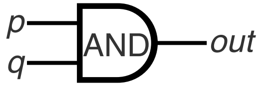
- take values 0/1 as inputs and produce 0/1 as output;
- correspond to logical connectives (many of them).

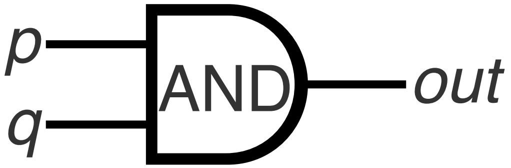
AND gate

AND connective

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

AND gate

p	q	
0	0	0
0	1	0
1	0	0
1	1	1



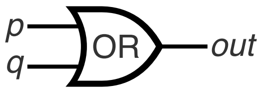
“Block looks like the D of an AND.”

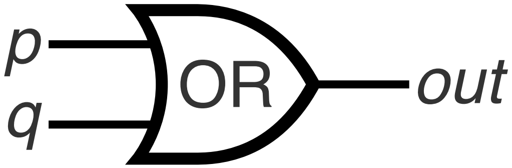
OR gate

OR connective

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

OR gate

p	q	
0	0	0
0	1	1
1	0	1
1	1	1



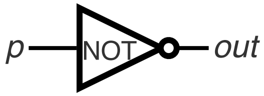
“Arrowhead block looks like ∨.”

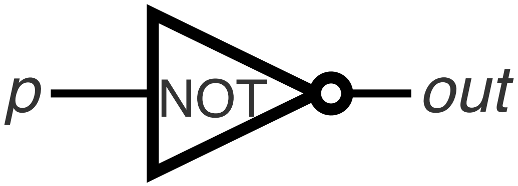
NOT gate

NOT connective

p	$\neg p$
F	T
T	F

NOT gate

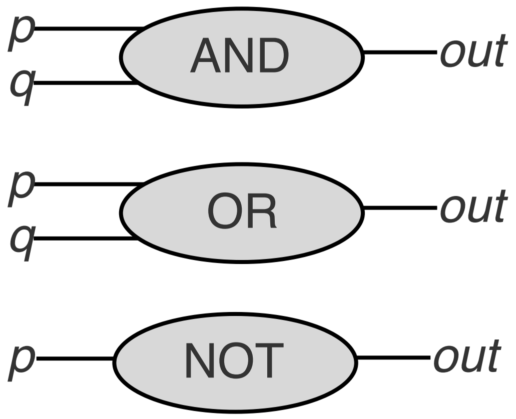
p	
0	1
1	0



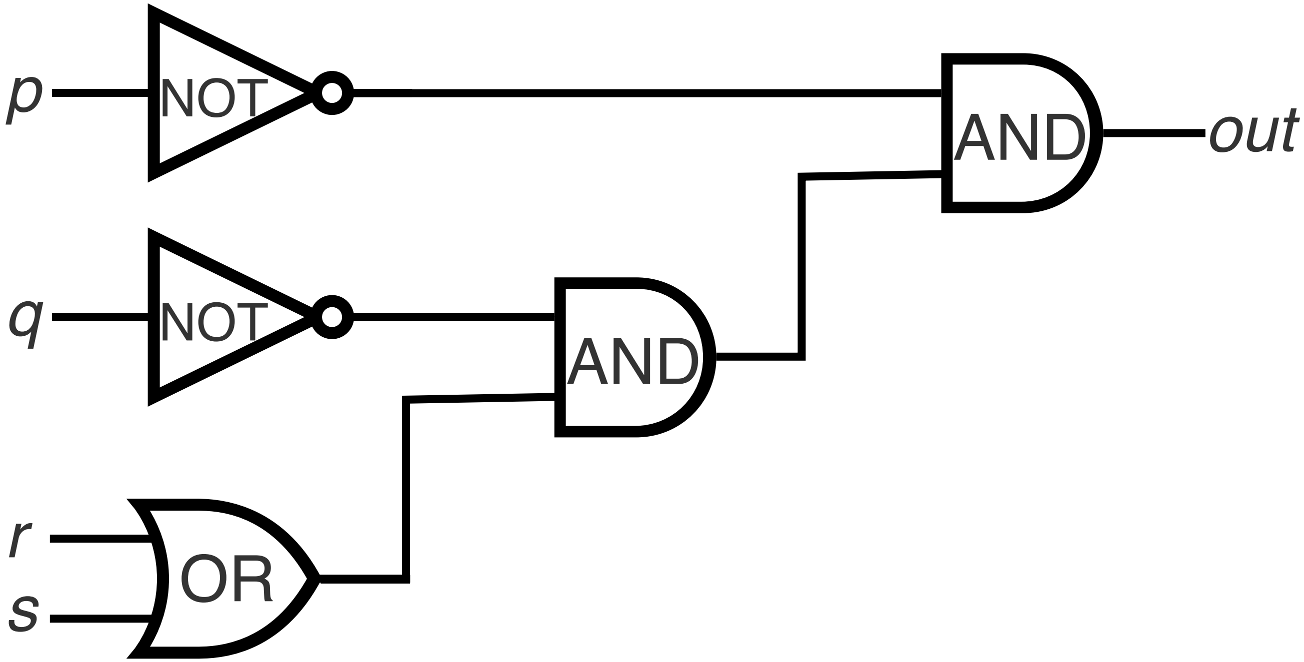
Also called an *inverter*.

Blobs are OK!

You may write gates using blobs instead of shapes.

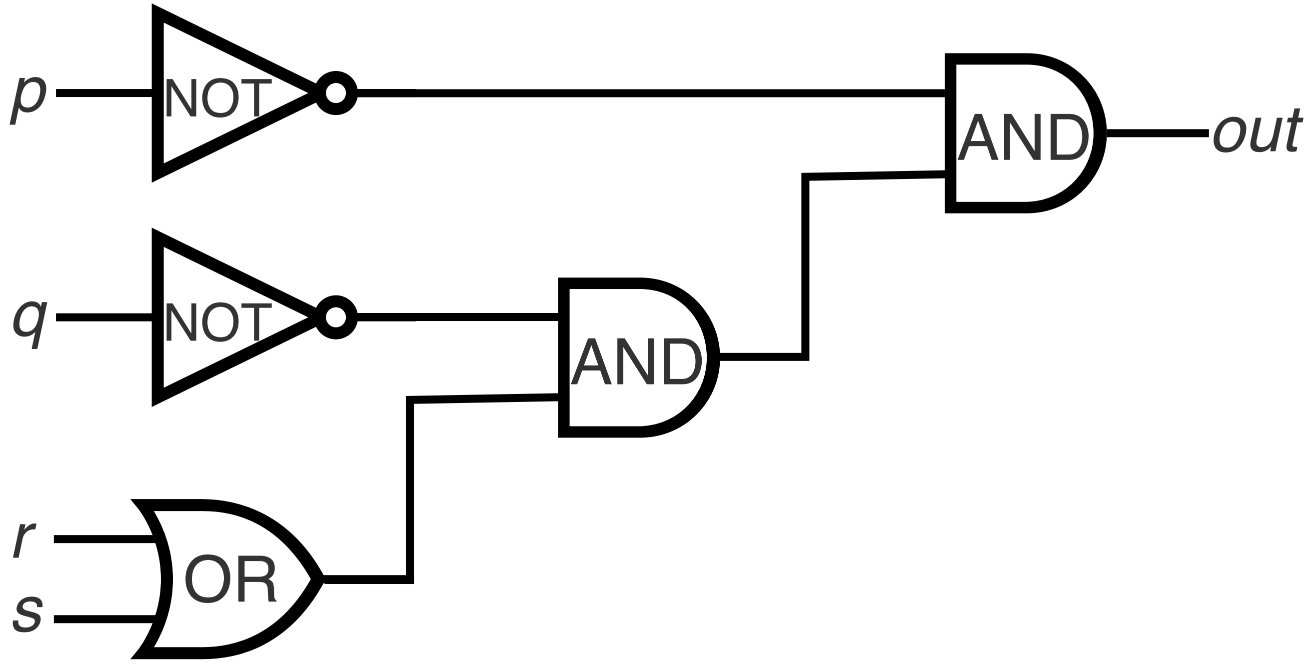


Combinational logic circuits: wiring up gates



Values get sent along wires connecting gates.

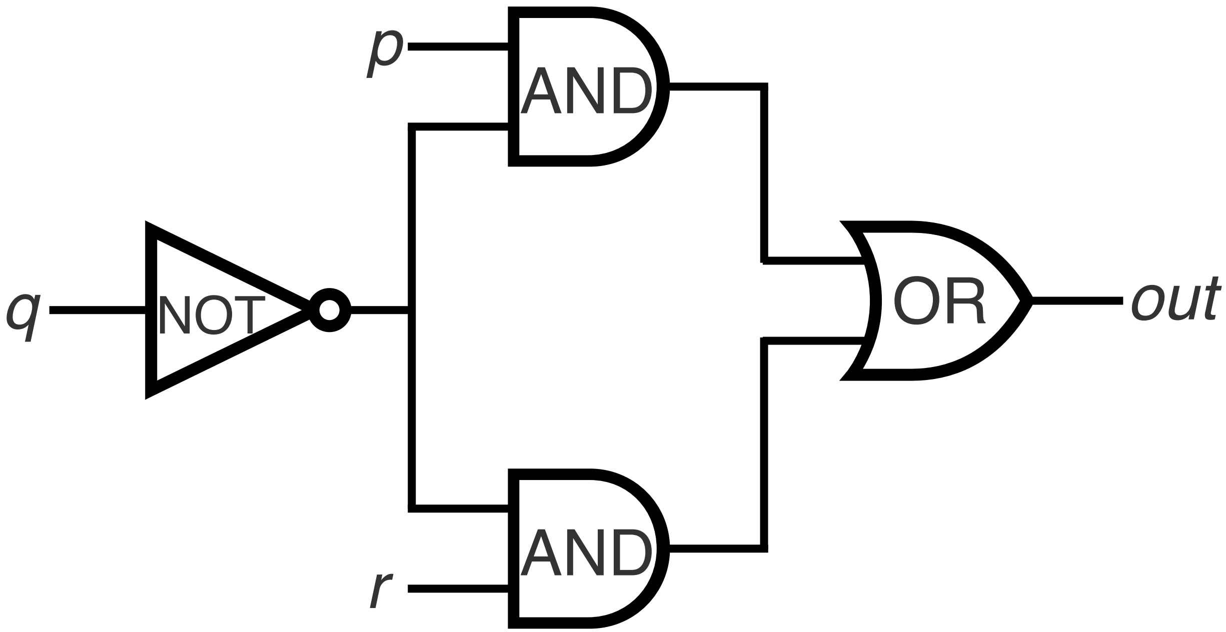
Combinational logic circuits: wiring up gates



Values get sent along wires connecting gates.

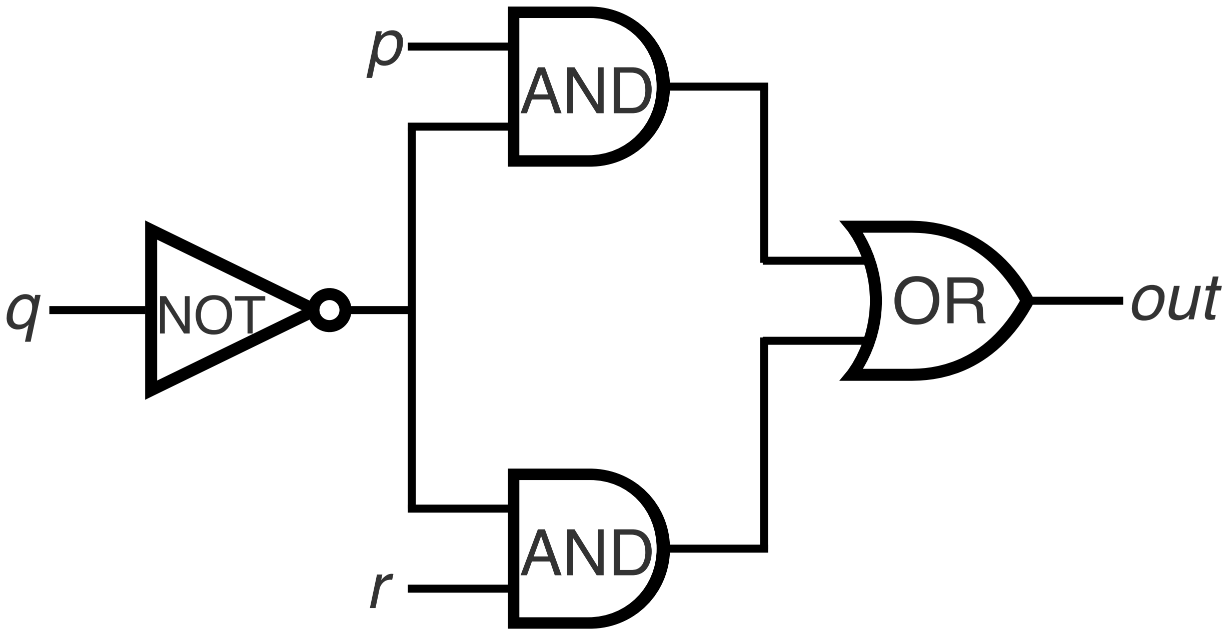
$\neg p \wedge (\neg q \wedge (r \vee s))$

Combinational logic circuits: wiring up gates



Wires can send one value to multiple gates.

Combinational logic circuits: wiring up gates



Wires can send one value to multiple gates.

$(p \wedge \neg q) \vee (\neg q \wedge r)$

Checking (circuit) equivalence

Describe an algorithm for checking if two logical expressions (or circuits) are equivalent.

What is the run time of the algorithm?

Why do we care?

Checking (circuit) equivalence

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Compute the entire truth table for both of them!

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What is the run time of the algorithm?

There are 2^n entries in the column for n variables.

Why do we care?

Checking (circuit) equivalence

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Compute the entire truth table for both of them!

What is the run time of the algorithm?

There are 2^n entries in the column for n variables.

Why do we care?

Program and hardware verification reduces to logical equivalence checking!

Summary

Propositions can be tautologies, contradictions, or contingencies.

Tautologies are always true.

Contradictions are never true.

Contingencies are sometimes true.

Propositions are equivalent when they have the same truth values.

Use truth tables or laws of logic to establish equivalence.

Digital circuits implement propositional logic!

F/T correspond to 0/1 (low/high voltage), respectively.

Gates implement logical connectives.

Combinational circuits implement compound propositions.