Homework 6 (due May 13 2020)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough to convince someone who does not already know the answer. You may use results from lecture and previous homeworks without proof. See the syllabus for more details and for permitted resources and collaboration.*

1. Alligator Eats The Bigger One (15 points)

Prove that for all integers n with $n \geq 1$, we have $n \cdot 6^n \leq (n+10)!$.

2. Running Time (20 points)

The running time of an algorithm as a function of the input size, $T(n)$, satisfies the following equations:

$$\begin{aligned} T(n) &= 5n && \text{if } 1 \leq n \leq 4 \\ T(n) &= T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + 5n && \text{for all } n > 4 \end{aligned}$$

Use strong induction to prove this *upper bound* on the running time: for all $n \geq 1$, we have $T(n) \leq 20n$.

Hint: The only fact about $\lfloor\ \rfloor$ that you will need is that when $x \geq 1$, $\lfloor x \rfloor$ is an integer and $1 \leq \lfloor x \rfloor \leq x$.

(This running time could arise, for example, if the algorithm operates on an input of size n by, first, calling itself recursively on an input of half that size; then, calling itself recursively on an input of quarter of that size; and finally, performing additional processing that takes $5n$ steps.)

3. Almost All the Strings (20 points)

Let 0^n stand for a string of n zeros. Let S be the set of strings defined as follows:

Basis Step: $0^3 \in S$; $0^7 \in S$;

Recursive Step: if $0^x, 0^y \in S$, then $0^x \bullet 0^y \in S$, where $\bullet$ stands for string concatenation.

Show that, for every integer $n \geq 12$, the set S contains the string 0^n.

Hint: Structural induction is *not* the right tool for solving this problem.

4. Apples-to-Apples (20 points)

Apples-to-Apples is a game where two players take turns removing apples from two bunches. We call the players player 1, the person who moved first at the start of the game, and player 2, the person who moved second. In a move, a player removes at least one apple from **only one** bunch (so a player cannot take apples from both bunches on their turn). The loser is the first player who is unable to remove any apples on their turn. So if there are no apples available at the start of a player's turn, they have lost the game.

Here is an examples of how a game with 2 apples in each bunch could be played from beginning to end.

Move 1 player 1: remove 1 apple from bunch 1
Move 1 player 2: remove 1 apple from bunch 1
Move 2 player 1: remove 2 apples from bunch 2
Player 2 cannot move, so player 1 wins.

(a) [15 points] Prove that player 2 can win any game of Apples-to-Apples **if both bunches contain the same number of apples**.

(b) [5 points] Describe the winning strategy of player 2. In other words, based on your proof in the previous part, explain how player 2 should always move in order to win the game. You don't need to prove anything, since you proved why this strategy works in the previous part.

5. Sets of sets of sets (25 points)

Let $t > 0$ be a positive integer and $T = \{3^0, \ldots, 3^{t-1}\}$ the set of the first t powers of 3. Define the set S to consist of all integers of the form $\sum_{a \in A} a + \sum_{b \in B} b$, where $A \in \mathcal{P}(T)$ and $B \in \mathcal{P}(A)$, i.e.:

$$S = \{n \in \mathbb{N} : \exists A. \exists B. A \in \mathcal{P}(T) \wedge B \in \mathcal{P}(A) \wedge n = \sum_{a \in A} a + \sum_{b \in B} b\}$$

Recall that $\sum_{x \in X} x$ stands for the sum of all the numbers in the set X.

Hint: You may use this fact without proof:

$$\sum_{i=0}^{n-1} ar^i = a\left(\frac{1 - r^n}{1 - r}\right) \text{ for } r \neq 1$$

(a) [5 points] What is the largest integer, $\max(S)$, in the set S? Prove that your answer is correct.

(b) [20 points] Prove that S consists of all the integers from 0 to $\max(S)$, i.e., $S = \{0, \ldots, \max(S)\}$.