Homework 4 (due April 29 2020)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough to convince someone who does not already know the answer. You may use results from lecture and previous homeworks without proof. See the syllabus for more details and for permitted resources and collaboration.*

1. A Proof Using Domain Properties (16 points)

Let the domain of discourse be the real numbers ($\mathbb{R}$). We define the predicate $\mathsf{OnLine}(a, b, x, y)$ to be true iff (x, y) lies on the line with slope a and intercept b (i.e., iff $ax + b = y$) and the predicate $\mathsf{OnCircle}(u, v, d, x, y)$ to be true iff $d > 0$ and (x, y) lies on the circle with diameter d and center at (u, v).

Give an English proof of the following claim:

$$\begin{aligned}\forall u.\forall v.\forall d.((d > 0) \rightarrow \exists a.\exists b.\exists x_1.\exists y_1.\exists x_2.\exists y_2.(((x_1 \neq x_2) \vee (y_1 \neq y_2)) \wedge \\ \mathsf{OnCircle}(u, v, d, x_1, y_1) \wedge \mathsf{OnCircle}(u, v, d, x_2, y_2) \wedge \\ \mathsf{OnLine}(a, b, x_1, y_1) \wedge \mathsf{OnLine}(a, b, x_2, y_2))\end{aligned}$$

2. Sets (18 points)

Prove each of the following claims for arbitrary sets A, B, and C.

(a) [8 points] $A \cap \overline{(B \setminus A)} = A$

(b) [10 points] $(B \setminus A) \cap (C \setminus A) \subseteq (B \cup C) \setminus A$

3. Power Sets (16 points)

Prove or disprove the following statements:

(a) [8 points] For any two sets S and T, it holds that:

$$\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T) \cup \mathcal{P}(S \cap T).$$

(b) [8 points] For any two sets S and T, it holds that:

$$\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T).$$

4. Cartesian Products (20 points)

(a) [15 points] Let A, B, and C be non-empty sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(b) [5 points] Prove that the Cartesian product is not commutative, i.e. Prove that it is **not true** that for any sets A, B that $A \times B = B \times A$. Hint: Use a proof technique from lecture.

5. Odd Congruences (10 points)

Prove that for any positive odd integer x, $(x^2 + 1) \equiv 2(\text{mod } 4)$.

6. Additive Inverses in $\mathbb{Z}_n$ (20 points)

Let $\mathbb{Z}_n$ denote the set of integers $\{0, 1, \cdots, n - 1\}$. Prove the following statements.

(a) [9 points] Every $a \in \mathbb{Z}_n$ has an additive inverse, i.e. prove for every $a \in \mathbb{Z}_n$ there is some $b \in \mathbb{Z}_n$ such that $(a + b) \equiv 0(\text{mod } n)$.

(b) [9 points] Every $a \in \mathbb{Z}_n$ has at most 1 additive inverse. Hint: use a proof technique from class.

(c) [2 points] Conclude using the previous parts that every $a \in \mathbb{Z}_n$ has a unique additive inverse.