Homework 3 (due April 22 2020)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough to convince someone who does not already know the answer. You may use results from lecture and previous homeworks without proof. See the syllabus for more details and for permitted resources and collaboration.*

1. From English to predicate logic and back (18 points)

For each of the following English statements, (1) translate it into predicate logic, (2) write the negation of that statement in predicate logic with the negation symbols pushed as far in as possible so that any negation symbols are directly in front of a predicate, and then (3) write a natural translation of (2) to English.

For the logic, let your domain of discourse be programs and values. You should use only the predicates $\mathsf{Accepts}(x, y)$ and $\mathsf{Returns}(x, y)$ which say that a program x accepts a value y as input or that x returns y as output; the predicates $\mathsf{Program}(x)$ and $\mathsf{Value}(x)$, which say whether x is a program or a value (respectively); and the predicates $x = y$ and $x \neq y$, which say whether x and y are the same object.

(a) [6 points] “Hello” is the only value that the program `HelloWorld` returns.

(b) [6 points] There is a value that is accepted by some program but returned by all programs.

(c) [6 points] Programs that return “Hello” don’t accept some value.

2. Spoof (16 points)

Theorem: Given $p \wedge \neg q$, $r \to s$, and $p \to \neg(\neg q \wedge s)$ prove $\neg r$.

“Spoof”:

1.	$p \wedge \neg q$	Given
2.	p	$\wedge$ Elim: 1
3.	$p \to \neg(\neg q \wedge s)$	Given
4.	$\neg(\neg q \wedge s)$	MP: 2,3
5.	$\neg\neg q \wedge \neg s$	DeMorgan’s: 4
6.	$q \wedge \neg s$	Double negation: 5
7.	$\neg s$	$\wedge$ Elim: 6
8.	$r \to s$	Given
9.	$\neg s \to \neg r$	Contrapositive: 8
10.	$\neg r$	MP: 7,9

(a) [6 points] What is the most significant error in this proof? Give the line and briefly explain why it is wrong.

(b) [10 points] Show that the theorem is true by fixing the error in the spoof.

3. Proof by cases (18 points)

In this problem, we will consider the following, new inference rule:

Proof By Cases

$$\frac{A \vee B \quad A \to C \quad B \to C}{\text{C}}$$

This rule says that, if we know that either A or B is true and that both A implies C and B implies C, then it follows that C is true. If it is A that is true, then we get that C is true by Modus Ponens and likewise if B is true instead. This is a valid rule of inference.

(a) [8 points] Use the Proof By Cases rule to prove the following. Given $p \wedge (q \vee r)$, $q \rightarrow (r \wedge s)$, and $r \rightarrow (r \wedge s)$, it follows that $p \wedge (s \vee t)$.

(b) [10 points] Prove that the "Elim $\vee$" rule follows from "Proof By Cases". Specifically, use the Proof by Cases rule to prove that, given $p \vee q$ and $\neg p$, it follows that q is true. (You may not use Elim $\vee$.)

4. Formal proofs in propositional logic (16 points)

(a) [8 points] Write a formal proof using inference rules that given $\neg(p \vee q) \rightarrow r$ and $\neg(p \vee r)$the proposition q must also be true.

(b) [8 points] Write a formal proof using inference rules that, given $p \oplus q$ and $s \oplus q$, the proposition $s \rightarrow p$ must also be true. You may use the additional equivalence $a \oplus b \equiv (a \vee b) \wedge (\neg a \vee \neg b)$, which we will call "Definition of $\oplus$".

5. Formal proofs in predicate logic (12 points)

Using the logical inference rules and equivalences we have given, write a formal proof that given $\forall x.(\exists y.\neg R(x,y)) \rightarrow P(x)$, $\forall x.Q(x) \rightarrow (P(x) \rightarrow R(x,x))$, and $\neg \exists x.R(x,x)$, you can conclude that $\exists x.\neg Q(x)$.

6. A formal proof and an English proof (20 points)

Recall that an integer n is even iff there exists an integer a such that $n = 2a$, and it is odd iff there exists an integer a such that $n = 2a + 1$.

(a) [12 points] Give a formal proof that, if n is odd and m is even, then $n + m$ is odd. In addition to the rules given in class, you can also rewrite algebraic expressions to equivalent ones using the rule "Algebra." (For example, you could write "$a(b + 1) - a = ab$" with Algebra as the rule / explanation.)

(b) [8 points] Write your proof from part (a) as an English proof.