Homework 6: Induction

CHANGELOG: This is Version 2 (uplaoded Saturday 11 AM) Question 7 had a typo in the definition of leaves. Because this is a find the bug problem, we will accept responses that refer to either the original definition or the corrected definition.

Due date: Wednesday November 25th at 11:59 PM (Seattle time, i.e. GMT-8)

If you work with others (and you should!), remember to follow the collaboration policy.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting.

Be sure to read the grading guidelines for more information on what we're looking for.

You **MUST** use induction for the proofs in this homework (unless the problem does not require a proof, or otherwise noted in the problem). You may use any appropirate version of induction (e.g. weak or strong or structural). Remember to define a predicate P as part of your proof.

1. Alligator Eats The Bigger One [20 points]

Prove that for all integers n with $n \geq 1$ we have $n \cdot 7^n \leq (n+13)!$.

2. Running Times [20 points]

You wrote a piece of recursive code. On an input of size n, your function takes $T(n)$ time to run, where:

$$\begin{aligned} T(n) &= 5n && \text{if } 1 \leq n \leq 4 \\ T(n) &= T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + 5n && \text{for all } n > 4 \end{aligned}$$

In the definition above, $\lfloor x \rfloor$ is the "floor" function, it returns the greatest integer at most x.
For example: $\lfloor 3.2 \rfloor = 3$, $\lfloor 3.7 \rfloor = 3$, $\lfloor 3 \rfloor = 3$.

Show that for all $n \in \mathbb{N}$ with $n \geq 1$, $T(n) \leq 20n$

Hint 1: Notice that while $T()$ is defined with equality, you are only proving an inequality.
Hint 2: The only fact about the floor function you will need is $\lfloor x \rfloor$ is an integer and $1 \leq \lfloor x \rfloor \leq x$.

3. str000ng induction would be a good choice [20 points]

Let 0^n mean a string of n zeros. Let S be the set of strings defined as follows:

Basis Steps: $0^3 \in S$, $0^5 \in S$
Recursive Step: If $0^x, 0^y \in S$ then $0^x \cdot 0^y \in S$ where $\cdot$ is string concatenation.

Show that, for every integer $n \geq 12$ the set S contains the string 0^n.

Caution: Structural Induction is not the best tool for this problem. Structural induction shows $\forall x \in S(P(x))$. You're analyzing what the elements of S are in this problem, not proving a predicate holds for all elements of S.

4. Apples-to-Apples [23 points]

The Apple Picking Game is played between two players, who take turns removing apples from two bunches. Player 1 moves at the start of the game and Player 2 moves second. In each move, a player chooses one of the two bunches, then removes at least one apple from the bunch (as many as they choose, from a minimum of one to a maximum of all remaining apples in that bunch). Note that a player cannot take apple(s) from both bunches in a single turn.

The loser is the first player who is unable to remove any apples on their turn. That is, if there are no apples remaining at the start of the player's turn, they have lost the game.

Here is an example of how a game is played. Initially there are 2 apples in both bunches.
Move 1, player 1: removes 1 apple from bunch A.
Move 1, player 2: removes 1 apple from bunch A.
Move 2, player 1: removes 2 apples from bunch B.
Move 2, player 2: There are no apples remaining. Player 2 loses.

(a) Prove that player 2 can win any game of the Apple Picking Game, if both bunches contain the same number of apples at the start of the game. [20 points]
Hint: There is more than one reasonable choice for $P()$ here, think carefully about how your $P()$ relates to how you can do your inductive step, and how you get the overall claim.

(b) Describe the winning strategy for player 2. In other words, based on your proof, explain how player 2 should move in order to ensure they will win the game. You do not have to prove anything for this part. [3 points]

5. The Apple Doesn't Fall Far From The... `Tree` [20 points]

In CSE 143, you saw a recursive definition of trees. That definition looks a little different from what we saw in class.

The following definition is analogous to what you saw in 143. We'll call them `JavaTrees`.

Basis Step: `null` is a `JavaTree`.
Recursive Step: If L, R are `JavaTrees` then $(\texttt{data}, L, R)$ is also a `JavaTree`.

Show that for all `JavaTrees`: if they have k copies of `data` then they have $k + 1$ copies of `null`.

Remark: You're effectively showing here that a binary tree with k nodes has $k + 1$ `null` child pointers.

6. Find. The. Bug. [7 Points]

Recall the definition of **Trees** we used in class:
Basis Step: $\bullet$ is a **Tree**.
Recursive Step: If L and R are **Trees**, then $\texttt{Tree}(\bullet, L, R)$ is a **Tree**.

And recall the following definition of height:
$\text{height}(\bullet) = 0$
$\text{height}\left(\texttt{Tree}(\bullet, L, R)\right) = 1 + \max\{\text{height}(L), \text{height}(R)\}$

And the definition of leaves:
(This definition, which appeared in the original upload, has a typo. You may build a response off this answer for full credit, but you may find it easier to use the correct definition)
$\text{leaves}(\bullet) = 1$
$\text{leaves}\left(\texttt{Tree}(\bullet, L, R)\right) = \text{leaves}(T) + \text{leaves}(R)$

And the (correct) definition of leaves:
$\text{leaves}(\bullet) = 1$
$\text{leaves}\left(\texttt{Tree}(\bullet, L, R)\right) = \text{leaves}(L) + \text{leaves}(R)$

Your friend wants to show $\forall$**Trees** T, $\text{leaves}(T) = 2^{\text{height}(T)}$.

Your friend decided to use strong induction (sturctural induction would have been a better choice). Here is their proof.

① Define $P(n)$ to be: "all **Trees** of height n have 2^n leaves". We show $P(n)$ for all $n \geq 0$ by induction on n.

② Base Case ($n = 0$)
Ⓐ Consider an arbitrary tree of height 0, there is only one such tree $\bullet$.
Ⓑ $\text{leaves}(\bullet) = 1 = 2^0 = 2^{\text{height}(\bullet)}$.

③ Inductive Hypothesis: Suppose $P(0) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 0$.

④ Inductive Step:
Ⓐ Let T_1 and T_2 be arbitrary **Trees** of height k. By IH applied to each, we have: $\text{leaves}(T_1) = 2^k$, $\text{leaves}(T_2) = 2^k$.
Ⓑ Define $T = \texttt{Tree}(\bullet, T_1, T_2)$.
Ⓒ To make a **Tree** of height $k + 1$, we must use the recursive rule.
Ⓓ Therefore, since T_1 and T_2 were arbitrary, T is an arbitrary **Tree** of its height.
Ⓔ $\text{height}(T) = 1 + \max\{k, k\} = k + 1$ and $\text{leaves}(T) = \text{leaves}(T_1) + \text{leaves}(T_2) = 2^k + 2^k = 2^{k+1}$
Ⓕ Since T is arbitrary **Tree** of height $k + 1$, and it has $2^{\text{height}(T)} = 2^{k+1}$ leaves, we have $P(k + 1)$

⑤ Therefore $P(n)$ is true for all $n \geq 0$ by the principle of induction.

⑥ Observe that every **Tree** has height at least 0, so we have for all **Trees** T, $\text{leaves}(T) = 2^{\text{height}(T)}$.

(a) The claim is false. Identify a counter-example. You should (1) draw (or otherwise describe) the example, (2) show that it is a **Tree**(e.g. by showing the rules to build it), (3) state its height and number of leaves. [3 points]

(b) Identify the biggest flaw in the proof. We have labeled the sentences to help you describe where it goes wrong. [4 points]